Lesson Overview

GOALS

- Find a recursive function that models a word problem.
- Write a rule for a recursive function.
- Work with recursive functions.

Students work with recursive functions in this lesson. As they saw in Investigation 5B, recursive functions are useful when fitting a function to a table. Here, they will see that recursive functions are also useful in word problems, in particular, when solving word problems involving interest.

If you have access to a computer lab with a spreadsheet program, this lesson can be a good time to show the mathematics of spreadsheets. You can complete this lesson using just calculators, but it may be beneficial when completing the interest exercises to encourage your students to keep track of their calculations in a table.

CHECK YOUR UNDERSTANDING
- Core: 1, 2
- Optional: 3a–d, 4
- Extension: 3e

MATERIALS
- computers with a spreadsheet program such as Excel (optional)
- graphing calculators
- Blackline Masters 5.12A, 5.12B

HOMEWORK
- Core: 5, 6, 7, 8, 9, 10, 11, 12a–b, 14, 15, 16
- Optional: none
- Extension: 12c–d, 13

VOCABULARY
- closed form
- exponential function
- fixed point
- Koch snowflake
- period
- periodic function

Launch

Begin this lesson by reviewing some of the work with recursive functions students learned in Investigation 5B. A good question might be:

If you invest $500 in a bank account, and that bank gives you 6% interest at the end of each year, how much money will you have after 5 years? After 10 years?

Explore

Minds In Action

Recursive functions as well as closed-form rules can describe the function in this example of paying off a debt with interest. Recursive functions describe this situation more easily. Students will explore how to find closed-form rules for this situation in a later course.

Also, this example shows how recursive functions can appear in everyday situations.

5.12 From Situations to Recursive Rules

In Lesson 5.9, you used this recursive function.

$$f(n) = \begin{cases} 3 & \text{if } n = 0 \\ f(n-1) + 5 & \text{if } n > 0 \end{cases}$$

When you read this function, use the word *or* to separate the two definitions, or cases, of the function. In words, the function of n equals 3 if n equals zero, or the function of n is given by the formula $f(n-1) + 5$ if n is greater than zero.

A key part of the notation is in the formula on the second line, $f(n-1) + 5$. You find the output for n by taking the previous output, $f(n-1)$, and adding 5. For example, you get $f(13)$ by adding 5 to $f(12)$.

The output values that this recursive function generates match the linear function $g(x) = 3 + 5x$, when you input nonnegative integers. Recursive functions can also define functions that are not matched by linear functions. A common recursive function that computes compound interest does not match a linear function.

The function $g(x) = 3 + 5x$ is in **closed form,** because you can write $g(x)$ as one expression.

Minds in Action episode 22

Louis and Andrea talk about their credit card bill.

Louis What's up with this bill? It adds up fast. We owe a thousand dollars.

Andrea We'd better cut up the card right now. At least our interest rate is only 11.99% APR. That's better than a lot of credit cards. We'll pay it down.

Louis The bill tells us that we can pay as little as 2% of what we owe each month. So 2% of $1000 is 20 dollars. That's cheap.

Andrea Yes, but not all of that is going to pay it down. They're basically charging almost 1% interest per month. So 10 dollars of the 20 dollars just goes to interest.

Louis Okay, so after a month we'll owe $990. Then it will even get a little easier, since we won't have to pay as much. Give me a few minutes.

Why is Andrea correct when she says that the interest is 1% per month? Andrea says that the annual rate is 11.99% and the bank calculates interest monthly. Each month the interest is one twelfth of 11.99%.

Louis builds this table on his computer.

Credit Card Payment: 2% Per Month

Month m	Starting Balance	Payment 2%	Interest 1%	Ending Balance
0	1000.00	(−) 20.00	(+)10.00	990.00
1	990.00	19.80	9.90	980.10
2	980.10	19.60	9.80	970.30
3	970.30	19.41	9.70	960.59
4	960.59	19.21	9.61	950.99
5	950.99	19.02	9.51	941.48
⋮	⋮	⋮	⋮	⋮
100	366.06	7.32	3.66	362.40

Louis Here, I built this payment plan using a spreadsheet. The numbers are rounded to the nearest cent, since they get messy. The spreadsheet shows how many months it will take to pay off the debt on the card. Let's look farther down.

Louis scrolls and scrolls.

Andrea Keep going. How far down are we now, 100 months?

Louis I know! Can you believe that? Even after 100 months we'll still owe more than a third of the money.

Andrea pushes a button to add all the entries in the payment column.

Andrea And we'll already have made over $1250 in payments. Only half of that amount went to pay down the principle. The other half went straight to the bank in interest.

Louis Well, what do we do? We can't just pay it off right now.

Andrea Maybe we can make more than the minimum payment. How does $50 per month sound?

Louis Alright, I guess. I'll change the spreadsheet.

Credit Card Payment: $50 Per Month

Month m	Starting Balance	Payment $50	Interest 1%	Ending Balance
0	1000.00	(–)50.00	(+)10.00	960.00
1	960.00	50.00	9.60	919.60
2	919.60	50.00	9.20	878.80
3	878.80	50.00	8.79	837.59
4	837.59	50.00	8.38	795.97
5	795.97	50.00	7.96	753.93
⋮	⋮	⋮	⋮	⋮
100	(5819.15)	50.00	(58.19)	(5927.34)

Louis So after 100 months, the bank owes us almost 6000 dollars?

Andrea No, no. It just means we've paid the whole thing off in less than 100 months. Get rid of all the rows with negatives. Here, month 22 is the last row.

In spreadsheets, accountants often display negative numbers by using parentheses instead of a negative sign. For example, (58.19) is the same as −58.19.

Credit Card Payment: $50 Per Month

Month m	Starting Balance	Payment $50	Interest 1%	Ending Balance
0	1000.00	(–) 50.00	(+)10.00	960.00
1	960.00	50.00	9.60	919.60
2	919.60	50.00	9.20	878.80
3	878.80	50.00	8.79	837.59
4	837.59	50.00	8.38	795.97
5	795.97	50.00	7.96	753.93
⋮	⋮	⋮	⋮	⋮
21	70.45	50.00	0.70	21.15
22	21.15	21.36	0.21	0.00

You can calculate the last payment by hand or program a computer to do the calculation

Andrea For that last month, all we do is pay what's left plus the tiny amount of interest for that month. So, the total we pay is $1121.35. That's 22 months at $50, plus the extra. We pay off the whole thing in less time and with much, much less interest.

Louis Wow. Well, let's try to stick to paying $50 per month then. It seems to work out really well.

For Discussion

1. Describe in words the rule that Louis and Andrea use to calculate each month's balance when they make the minimum monthly payments.

   How does the rule change when the credit card charges 15% interest instead of approximately 12%?

Louis follows a rule when he sets up the spreadsheet for minimum payments. You can write the rule in function form. The balance B is a function of the month n.

$$B(n) = \begin{cases} 1000 & \text{if } n = 0 \\ B(n-1) - 0.02 \cdot B(n-1) + 0.01 \cdot B(n-1) & \text{if } n > 0 \end{cases}$$

In this situation, what do the three expressions $B(n-1)$, $-0.02 \cdot B(n-1)$, and $+0.01 \cdot B(n-1)$ represent?

You can use the Distributive Property to simplify the rule. Remember, $B(n-1)$ is the output of the function B when you input $n-1$. $B(n-1)$ represents a single value.

When the function notation is confusing, you can think of $B(n-1)$ as a value, such as x. You can replace the expression $B(n-1)$ with x.

$$x - 0.02x + 0.01x$$

In this expression, it is simpler to see that you can combine the like terms to get $0.99x$.

So, a simplified way to write the function is the following.

$$B(n) = \begin{cases} 1000 & \text{if } n = 0 \\ 0.99 \cdot B(n-1) & \text{if } n > 0 \end{cases}$$

To model this function on your calculator, see the Technology Handbook.

Like the function in Lesson 5.10, this function is recursive. The difference is that instead of adding a value to the previous output, you multiply the previous output by a value. A function like the one above is not linear. It is an **exponential function.**

Example 1

Problem Melissa's doctor prescribes amoxicillin for two weeks to cure an ear infection. She starts without any amoxicillin in her body. Melissa takes 1.5 grams of amoxicillin each day. Her body will consume one fourth of the amoxicillin each dose contains each day. Write a recursive function to find how much amoxicillin is present at the end of two weeks of treatment.

Solution To find this function, start by calculating the amount of amoxicillin for a few days.

Answers

For Discussion

1. Starting Balance − 2% of Balance + 1% of Balance = Ending Balance; Balance − 2% of Balance + 1.25% of Balance = Ending Balance

For Discussion

PROBLEM 1 It is important for students to understand that getting from one month's balance to the next month's balance involves combining a few different calculations.

Written as a recursive function, the balance at the end of month n can be expressed as $B(n) = B(n-1) - (0.02 \cdot B(n-1)) + (0.01 \cdot B(n-1))$. If the company charged 15% interest, then each month you would add $\frac{0.15}{12}$, which equals 0.0125 or 1.25%. The recursive function that represents this situation is $B(n) = B(n-1) - (0.02 \cdot B(n-1)) + (0.0125 \cdot B(n-1))$.

Example 1

This example is another problem that illustrates the usefulness of recursive functions in everyday situations.

Amoxicillin Treatment

Day	Amoxicillin (g)	How to Find Amoxicillin Amount
0	0	This is the amount on the day before treatment begins.
1	1.5	This is the amount of the first day's pills.
2	2.625	Add second day's pills (1.5 g). Then subtract $\frac{1}{4}$ of previous total (0.375 g).
3	3.469	Add third day's pills (1.5 g). Then subtract $\frac{1}{4}$ of previous total (0.656 g).
4	4.102	Add fourth day's pills (1.5 g). Then subtract $\frac{1}{4}$ of previous total.
5	4.576	Repeat the process you used for Day 4.

Notice that you are doing the same thing again and again to get from one output to the next. This is exactly what a recursive function does. The amount of amoxicillin A depends on the day d. The function rule is the following.

$$A(d) = \begin{cases} 0 & \text{if } d = 0 \\ A(d-1) + 1.5 - \frac{1}{4}A(d-1) & \text{if } d > 0 \end{cases}$$

You can combine the like terms $A(d - 1)$, which means that you can use the expression $0.75A(d - 1) + 1.5$ in a spreadsheet program or calculator.

To model this function on your calculator, see the Technology Handbook.

For You to Do

2. Repeat the function in the example for nine more days. How much amoxicillin is in Melissa's body at the end of 14 days?
3. Plot the data for the first 14 days. What does your graph look like?

For Discussion

4. Suppose Melissa continues taking amoxicillin for another two weeks. What will happen to the amount of amoxicillin in her body? Is there an amount of amoxicillin that Melissa could take that would not change the amount in her body from day to day?
5. Suppose Melissa stops taking amoxicillin after 14 days. What happens to the level of amoxicillin in her body after the last day? Extend your graph for the third week.

This amount of amoxicillin represents a **fixed point,** since its output equals its input.

Recursive functions repeat different operations. In this lesson, you have used recursive functions that repeat addition and multiplication. Recursive functions can also produce other patterns.

Answers

For You To Do

2. 5.955 g
3. See graph for 14 days in Problem 5.

For Discussion

4. It will get closer and closer to 6 g. Yes; taking 0 g each day will leave the amount unchanged. Once started, however, no.
5. See back of book.

Example 2

Problem The first two outputs for a function are 5 and 8. The recursive function is the following. To find an output, take the previous output and subtract the one before it. Find the 100th output.

Solution Since this recursive function does not seem to be linear or exponential, use the function to find outputs until a pattern emerges.

Function Outputs

Input	Output	How to Find Output
1	5	The value is given.
2	8	The value is given.
3	3	Take the previous output (8) and subtract the output before the previous output (5).
4	−5	Take the previous output (3) and subtract the output before the previous output (8).
5	−8	Repeat the process from Day 4, (−5) −3.
6	−3	Repeat the process from Day 5, (−8) − (−5).
7	5	(−3) − (−8)
8	8	5 − (−3)

The outputs seem to repeat. The eighth output is the same as the second one. When you extend the table, the 14th output is the same as the eighth one. The pattern continues. A function that repeats like this is a **periodic function.** This table shows the inputs that give a certain output value.

See the Technology Handbook to model these functions on your calculator.

Function Inputs and Outputs

Output	Inputs
5	1, 7, 13, 19, 25, . . . , 97, 103, . . . , $6n + 1$
8	2, 8, 14, 20, 26, . . . , 98, 104, . . . , $6n + 2$
3	3, 9, 15, 21, 27, . . . , 99, 105, . . . , $6n + 3$
−5	4, 10, 16, 22, 28, . . . , **100**, 106, . . . , $6n + 4$
−8	5, 11, 17, 23, 29, . . . , 101, 107, . . . , $6n + 5$
−3	6, 12, 18, 24, 30, . . . , 96, 102, . . . , $6n + 6$

The 100th output is −5. Note that 96 is the largest multiple of 6 that is not more than 100. It takes 4 more steps to go from the 96th output to the 100th.

For Discussion

6. What is the 200th output? The 300th output?

For Discussion

6. 8; −3

Example 2

This example is considered optional. It is an interesting example of the strange behavior some recursive functions exhibit, but it is not essential.

For Discussion

The outputs occur in a pattern of 6. So, to find the 200th output and the 300th output, students can determine how many more than a multiple of 6 each number is. Then, they can find the corresponding output in the table. 200 is two more than a multiple of 6, 198. 300 itself is a multiple of 6.

In-Class Experiment

PROBLEM 7 Blackline Master 5.12A provides a blank table for students to complete.

Wrap Up

Use the In-Class Experiment to wrap up the lesson. Discuss each group's results as a whole class. Take this opportunity to discuss how interest, although it seems like a very small percent, can add up and really grow to become a significant amount of money.

Make sure when assigning Problem 7 that you assign only one part to each student or small group. Otherwise, this problem can be time-consuming. You may want to be careful in student choice when assigning the $10 payment option. The credit card is never paid off under this option.

Use of a spreadsheet program is by far the preferred option in this problem, since it can add the payments and interest very quickly. If spreadsheets are not available, you might assign only those entries in the table that can be computed with less work (payments of $40 and higher).

Assessment Resources

Lesson Quiz 5.12

1. Use the function rule for parts (a) and (b).

$$C(n) = \begin{cases} 4, & \text{if } n = 0 \\ 2C(n-1), & \text{if } n > 0 \end{cases}$$

a. Find $C(4)$. **b.** Is this an exponential rule? Explain.

2. Each workday Bob drives to and from work on a toll road. The tolls total $3.25 each way. Bob purchases a pass containing $100 that allows him to pay the toll electronically as he drives through the tollbooth.

a. Make a table for the amount of money that remains on Bob's pass after n workdays, where n is an integer from 0 to 5.

b. Write a recursive rule for $B(n)$, the amount of money that remains on Bob's pass after n workdays.

c. Use the rule to find out how many days Bob can drive to work before he needs to refill the electronic pass.

3. Find a closed-form rule for the following recursive rule.

$$f(x) = \begin{cases} -1, & \text{if } x = 0 \\ f(x-1) + 3, & \text{if } x = 1 \end{cases}$$

In-Class Experiment

In episode 22, Louis and Andrea pay $50 per month on their credit card balance. They pay a total of $1121.35 on a $1000 balance. To find the total that they pay, they use the recursive function

$$B(n) = \begin{cases} 1000 & \text{if } n = 0 \\ B(n-1) - 50 + 0.01B(n-1) & \text{if } n > 0 \end{cases}$$

until the balance is below $50. Then they pay off the entire balance (payment plus interest) the next month. This process takes 22 months.

Suppose that instead of paying $50 per month, Louis and Andrea pay a different amount. In this experiment, you will look at different monthly payments. You will see how a different monthly payment affects the total amount they have to pay and the number of months they take to pay off the balance.

7. Use a calculator or computer. Each group in the class chooses a different monthly payment amount in the table. Find how many payments you have to make, what the total payment is, and how much of the total payment each month is interest. Complete the table by collecting the results from each group.

Payment Options: $1000 Balance at 12% Interest

Payment	Months to Pay	Total Payments	Interest
$10	■	■	■
$15	■	■	■
$20	■	■	■
$25	■	■	■
$30	■	■	■
$40	■	■	■
$50	22	$1121.35	$121.35
$60	■	■	■
$75	■	■	■
$100	■	■	■

a. $10 **b.** $15 **c.** $20 **d.** $25 **e.** $30
f. $40 **g.** $60 **h.** $75 **i.** $100

Answers

In-Class Experiment

7. a. $10: not large enough; will not even pay the interest.

b–i. Shown below are the amount, number of payments, total payments, and total interest. Values may vary by small amounts due to rounding.

b. $15: 111; $1656.31; $656.31
c. $20: 70; $1393.23; $393.23
d. $25: 52; $1283.49; $283.49
e. $30: 41; $1222.52; $222.52
f. $40: 29; $1156.50; $156.50
g. $60: 19; $1099.47; $99.47
h. $75: 15; $1078.68; $78.68
i. $100: 11; $1058.98; $58.98

Exercises Practicing Habits of Mind

Check Your Understanding

1. Describe this function in words. Then use the function to find $F(n)$ for the values of n from 1 to 12.

$$F(n) = \begin{cases} 1 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ F(n-1) + F(n-2) & \text{if } n > 2 \end{cases}$$

2. Some banks offer rainy-day savings accounts. You can send part of your paycheck directly to this sort of savings account. The money earns a small amount of interest every month. Kara opens a rainy-day account at Fourth Seventh Bank that offers 1.5% interest annually. The bank splits the interest payments into twelve months.

Assume that the bank does not give interest on money that you deposit this month. The bank pays interest only on the amount already in the account.

a. What percent interest does Kara earn each month from this account? How do you write the percent interest as a decimal? Describe your process in a way that you can use for any interest rate.

b. Kara deposits \$50 into this account every month. How much interest does she earn in the first month? In the second month? In the third month? You can use this table to help you.

Kara's Account Balance

Month	Starting Balance	Interest	Deposit	Ending Balance
0	\$0	\$0	\$ 0	\$0
1	\$0	■	\$50	■
2	■	■	\$50	■
3	■	■	\$50	■

c. Write a recursive function to find Kara's account balance after n months.

3. You can apply recursive functions to geometric shapes. One such shape is the **Koch snowflake.** The beginning shape in a Koch snowflake is an upside-down equilateral triangle. Then the snowflake develops following a recursive function.

Take every line segment in the shape and replace the middle third with two sides of an equilateral triangle pointing outward.

Exercises

1. For inputs 1 and 2, outputs are 1.For any other input, the output is the sum of the two previous outputs. 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144

2. a. 0.125% each month; 0.00125; divide the annual interest rate by 12.

b. \$0; \$0.06; \$0.13

c. $S(n) = 0$ if $n = 0$,
$S(n) = S(n-1) + 0.00125$
$S(n-1) + 50$ if $n > 0$

Exercises

HOMEWORK
- Core: 6, 7, 8, 9, 10, 11, 12a–b, 14, 15, 16
- Optional: none
- Extension: 12c–d, 13

Check Your Understanding

EXERCISE 1 It is extremely likely students will have seen the Fibonacci numbers in some form before this course. The exercise exists here mainly to give students practice with the notation. It is the first instance when $F(n-2)$ is being used.

EXERCISE 2 Blackline Master 5.12B provides a blank table for students to complete.

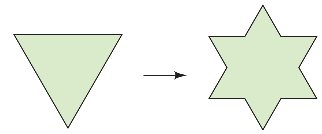
Students may need a reminder that a percent is out of 100, so the decimal point should be shifted two places when writing the percent as a decimal. Rewriting percents as decimals is a common stumbling point for students. An interest rate of 0.125% is often mistakenly executed as if it is 1.25% or even 12.5%. When the percent is in decimal form, students may not even realize the mistake.

EXERCISE 3 is a first introduction to fractal patterns, which students will explore closely in CME Project—*Geometry*. Push strong students to complete Take It Further part (e). You may want to lead a discussion about how the Koch snowflake has infinite perimeter (in the long run) but finite area (1.6 square units).

EXERCISE 4 This exact same problem will reappear in Chapter 6 when students look at tables with constant differences and again in Chapter 8 when students look at difference tables for quadratics. It is fine to stop here with students realizing that the function is not linear, but if students want to explore further, encourage them to do so.

For example, using this function once produces the level 1 snowflake.

Level 0 **Level 1**



The original triangle is the level 0 snowflake, even though it does not look like a snowflake.

In the level 2 snowflake, you transform the twelve line segments in the level 1 snowflake using the same function.

a. Draw a level 2 snowflake.

b. Suppose the original triangle has a side length of 1 and a perimeter of 3. Find the perimeters of the level 1 and level 2 snowflakes.

c. Find a recursive function $P(\ell)$ for the perimeter where ℓ is the level number. Is this function linear? Is it exponential?

d. Suppose you change the triangle's size so that the triangle has an area of 1 square unit. Find the areas of the level 1 and level 2 snowflakes.

e. **Take It Further** Find a recursive function for the area $A(\ell)$, where ℓ is the level number. Then find the area of the level 6 snowflake. What happens when you apply the function to further levels of the snowflake?

4. The output for a function starts with zero. Then the function rule is "Add 1, add 2, add 3, add 4, and so on." The recursive function for this rule is the following.

$$C(n) = \begin{cases} 0 & \text{if } n = 0 \\ C(n-1) + n & \text{if } n > 0 \end{cases}$$

a. Find the value of $C(5)$ by repeating the function.

b. Is this function linear? Explain.

On Your Own

5. Melissa's doctor tells her that, instead of taking three amoxicillin pills per day, she needs to take only two pills.

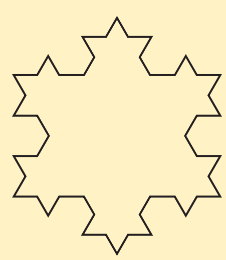
a. When there is 1.5 grams of amoxicillin in 3 pills, how much amoxicillin is there in 2 pills?

b. It is still true that one fourth of the amoxicillin will not be in her body the next day. How does the change in Melissa's prescription affect the recursive function shown below, which you used in Example 1?

$$A(d) = \begin{cases} 0 & \text{if } d = 0 \\ A(d-1) + 1.5 - \frac{1}{4}A(d-1) & \text{if } d > 0 \end{cases}$$

This exercise is similar to the example in this lesson.

3. a.



b. 4, $\frac{16}{3}$

c. $P(\ell) = 3$ for $\ell = 0$, $P(\ell) = \frac{4}{3}P(\ell - 1)$ for $\ell > 0$; exponential

d. $\frac{4}{3}, \frac{40}{27}$

e. $A(\ell) = 1$ for $\ell = 0$, $A(\ell) = A(\ell - 1) + \frac{3}{4}\left(\frac{4}{9}\right)^{\ell}$ for $\ell > 0$; about 1.5954; the area seems to get closer and closer to 1.6.

4. a. 15

b. not linear because the differences are not constant

5. a. 1 g

b. The 1.5 changes to 1.

c. 3466 mg

c. Suppose Melissa takes 2 pills of amoxicillin per day. Find the amount of amoxicillin in her body after one week, to the nearest milligram. She starts without any amoxicillin in her body.

6. Each weekday, Cheng travels on the Metro Rail System making a round trip from Greenbelt to Metro Center. The daily fare is \$2.85 each way. Cheng purchases a \$100 fare card to pay the \$2.85 fare for each ride.

   a. Make a table that shows the amount of money Cheng has left on the fare card after n days. Show the values of n from 0 to 5.

   b. Describe a recursive function for the situation, using words.

   c. Write the recursive function $B(n)$, the amount of money Cheng has left on his fare card after n workdays.

   d. Is this function linear? Is it exponential?

   e. Use the function to find how many days Cheng can ride the Metro to and from work before he needs to add money to the fare card.

7. **Write About It** In Example 2, which asks for the 100th output, the recursive function tells you to take the previous output and subtract the output before the previous output. The function gives the values 5 and 8 as the first and second outputs. Explain why this function definition matches the description above.

$$S(n) = \begin{cases} 5 & \text{if } n = 1 \\ 8 & \text{if } n = 2 \\ s(n-1) - s(n-2) & \text{if } n > 2 \end{cases}$$

The expression $s(n - 1)$ represents the previous output. What does the expression $s(n - 2)$ represent?

For Exercises 8 and 9, use the recursive function T.

$$T(n) = \begin{cases} 5 & \text{if } n = 1 \\ 8 & \text{if } n = 2 \\ T(n-1) - T(n-2) & \text{if } n > 2 \end{cases}$$

8. Find the 374th output.

9. In Function T, the outputs repeat with a period of 6. Suppose the first output is y, the second output is z, and the third output is $z - y$.

   a. Using algebra, find the sequence of the first 6 outputs.

   b. What does using variables to find the sequence of the first 6 outputs tell you about the period and the relationships between numbers in the sequence?

The **period** is the length of a repeating sequence. The sequence 7, 8, 9, 7, 8, 9, . . . has a period of 3.

10. **Standardized Test Prep** Which of the following is a closed form of the recursive function $T(n)$?

$$T(n) = \begin{cases} 0 & \text{if } n = 0 \\ T(n-1) + \frac{n}{2} & \text{if } n > 0 \end{cases}$$

A. $n - \frac{1}{2}$ **B.** n^2 **C.** $\frac{n^2 + n}{2}$ **D.** $\frac{n^2 + n}{4}$

On Your Own

EXERCISE 6 is, in some ways, a review of the first two investigations in this chapter. The only new material here is the written recursive definition, which should be the focus of any major discussion. Part (e) presents a bit of a trick, since Cheng can get to work on the 18th day (but he cannot return). Many students will answer 17 here, and in a way they are correct. You should check that students answering 18 are not just rounding to the nearest integer, and that they understand the situation.

EXERCISE 7 is the second instance of a function involving the input $n - 2$, so if students are stumbling, refer them to Exercise 1 in Check Your Understanding. They should be all right if they understand that $s(n - 1)$ represents the previous output.

EXERCISES 8 AND 9 Students' most likely errors here are in subtraction of negative numbers. For example, $(-8) - (-5) = -3$ is a frequently missed step. Students working in groups are likely to correct this error themselves, but watch out for it. Some students may not recognize this as the same function as in Exercise 7 because of the change in the function name, but this mistake is more rare.

EXERCISE 9 follows up on what students might see in Exercise 8. It is a generalization, so it may be on the hard side for some students, but it is proof that there is a pattern in the outputs.

6. a. (0, \$100), (1, \$94.30), (2, \$88.60), (3, \$82.90), (4, \$77.20), (5, \$71.50)

b. Sample: Value at day 0 is \$100. The value at the end of each day is \$5.70 less than the value at the end of the previous day.

c. $B(n) = 100$ if $n = 0$; $B(n) = B(n - 1) - 5.7$ for $n > 0$

d. linear

e. Cheng can ride the Metro to and from work for 17 days. On the 18th day, he will have to add money to the fare card in order to get home.

7. Check students' work.

8. 8

9. a. $y, z, z - y, -y, -z, y - z$

b. The pattern repeats with a period of 6.

10. D

EXERCISE 11 Students have probably played with their calculators in this way before, but perhaps not with the square root button. Some other explorations here include adding a value, multiplying by a value, the COS and SIN buttons, and the LOG button.

EXERCISE 12 previews later material on the Mandelbrot fractal. This recursive function is the exact recursive function used to make the Mandelbrot fractal—for now, *N* is restricted to be a real number, but the functions are the same. Parts (c) and (d) are truly difficult. You should assign them only to advanced students or classes.

Additional Resources

PRINTED RESOURCES
- Texas Instruments Activities Workbook
- Cabrilog Activities
- Teaching Resources
- Practice Workbook
- Assessment Resources

TECHNOLOGY
- TeacherExpress CD-ROM
- **Exam*View*** CD-ROM
- **PHSchool.com**
  - Homework Help
  - Video Tutors
  - Multiple Choice
  - Crosswords

Additional Practice

A tank holds 40,000 gallons of water. Twice a day, consumers use one eighth of the amount of water that was in the tank at the beginning of that day.

1. Make an input-output table. Calculate the number of gallons of water that remain in the tank at the end of each day from Day 0 to Day 8. Record the amounts to the nearest gallon.
2. Describe a recursive rule for the situation, using words.
3. Write the recursive rule $W(n)$, the number of gallons of water in the tank after n days.
4. Is this function linear? Is it exponential? Write the rule in closed form.
5. Assuming the pattern continues, when will the amount of water in the tank be less than 1 gallon?

For Exercises 6 and 7, use recursive rules 1 and 2.

Rule 1:

$$r(n) = \begin{cases} 10, & \text{if } n = 1 \\ 5, & \text{if } n = 2 \\ r(n-1) - r(n-2), & \text{if } n > 2 \end{cases}$$

Rule 2:
The first two outputs are 10 and 5. To find each subsequent output, take the previous output and subtract the output before it.

6. Find the first 10 outputs for Rules 1 and 2.
7. Explain why the two rules represent the same function.
8. Explain how to find the 155th output for the function $T(x)$.

$$T(x) = \begin{cases} -5, & \text{if } x = 1 \\ 4, & \text{if } x = 2 \\ T(x-1) - T(x-2), & \text{if } x > 2 \end{cases}$$

9. Use the function definition $G(y) = \begin{cases} N, & \text{if } y = 0 \\ 2[G(y-1)]^2 + N, & \text{if } y > 0 \end{cases}$.
   a. Describe the function using sentences.
   b. What is the output in the long run when you start with $N = 2$? With $N = 0$? With $N = -1$?

For Exercises 10–13, find a closed-form function that gives the same outputs as each recursively defined function.

10. $f(n) = \begin{cases} -5, & \text{if } n = 0 \\ f(n-1) + 2, & \text{if } n > 0 \end{cases}$
11. $h(n) = \begin{cases} -7, & \text{if } n = 0 \\ h(n-1) - 1, & \text{if } n > 0 \end{cases}$
12. $g(n) = \begin{cases} 8, & \text{if } n = 0 \\ g(n-1) - 3, & \text{if } n > 0 \end{cases}$
13. $k(n) = \begin{cases} -q, & \text{if } n = 0 \\ k(n-1) + p, & \text{if } n > 0 \end{cases}$

Practice: For Lesson 5.12, assign Exercises 1–13.

11. If you have ever been bored, you might have started pressing the same calculator button over and over again. Now try entering the largest number that fits on your calculator screen. Then press the square root button again and again.

a. Explain why this recursive function describes what you are doing. What do the letters b, N, and X represent?

$$X(b) = \begin{cases} N & \text{if } b = 0 \\ \sqrt{X(b-1)} & \text{if } b > 0 \end{cases}$$

b. What output do you get eventually? Will you get the same result no matter what number you enter first?

12. Benoit shows you a recursive function with some unpredictable behavior. Pick a number N and use this function definition.

$$Y(b) = \begin{cases} N & \text{if } b = 0 \\ (Y(b-1))^2 + N & \text{if } b > 0 \end{cases}$$

The function tells you to add N, not $Y(n - 1)$.

a. Describe this function using sentences.

b. What is the output in the long run when you start with $N = 1$? With $N = 0$? With $N = -1$?

c. Take It Further What starting values for N give outputs that approach infinity? What starting values for N give outputs that do not approach infinity? Prove your results.

d. Take It Further The starting value -1 repeats in a cycle of period 2. This value is the only number with this period. Find a number that repeats in a cycle of period 3.

Maintain Your Skills

Find a closed-form function that gives the same outputs as each recursively defined function.

13. $f(n) = \begin{cases} 3 & \text{if } x = 0 \\ f(n-1) + 4 & \text{if } x > 0 \end{cases}$

14. $g(n) = \begin{cases} 7 & \text{if } x = 0 \\ g(n-1) + 5 & \text{if } x > 0 \end{cases}$

15. $h(n) = \begin{cases} 1 & \text{if } x = 0 \\ g(n-1) - 5 & \text{if } x > 0 \end{cases}$

16. $r(n) = \begin{cases} b & \text{if } x = 0 \\ r(n-1) + a & \text{if } x > 0 \end{cases}$

Answers

11. a. b is the number of times you push the square root button. N is your starting number. It has to be positive. S is the value on the screen after you push the square root button b times.

b. 1; you get the same result no matter what positive number N you begin with.

12. a. Begin with any number. For any other input, square the previous output and add the original number.

b. For $N = 1$, the outputs get large very quickly. For $N = 0$, the outputs are all 0. For $N = -1$, the outputs alternate between 1 and 0.

c. Any value less than -2 or greater than 0 gives outputs that approach infinity. Values from -2 to 0 give outputs that do not approach infinity.

d. approximately -1.755

13. $n \to 4n + 3$

14. $n \to 5n + 7$

15. $n \to -5n + 1$

16. See Back of book.

Chapter Project

This project is optional. It includes exercises that require spreadsheet software. There are more exercises than you may want to assign to your class. Assign exercises to advanced students, or let them choose which problems interest them the most.

You will need to do these exercises in a computer lab where the computers have a spreadsheet program. You may want to allow time to review how to use a spreadsheet program with your class.

Exercise 1 is intended for the entire class to complete. If students are able to complete this exercise, they should be familiar enough with the spreadsheet program to work on any of the other exercises in this project.

For more help using spreadsheet software, visit PHSchool.com and reference the Web Code on p. 495.

EXERCISE 1 Students' most common errors here are in spreadsheet use. Mistakes about percent-to-decimal conversion are less likely here, but may still happen. Keep an eye out for these mistakes.

EXERCISE 2 If you are going to assign this exercise, be sure to ask students to gather the information on savings and CD accounts prior to when you intend to work on this project.

EXERCISE 4 practices proportional reasoning. Students may not notice the proportion at first, but when they divide one payment by the other, it should be obvious.

EXERCISE 5 If students use the same spreadsheet as in Exercise 3, make sure they do not forget to apply the change in interest rate for each month.

Project Using Mathematical Habits

Managing Money

Do you ever dream of going to college or buying a car? Usually, you save and borrow money for these large expenses. In this project, you will write a recursive rule to use in a spreadsheet and make decisions about managing money.

This project explores three ways to save and invest money: a certificate of deposit (CD), a savings account, and stock. In each case, you earn interest by lending your money. For each type of investment, the Annual Percentage Rate (APR) describes the investment's growth each year.

Materials

Paper, pencil, spreadsheet software

Earning Interest

1. Kara looks at three options for making her annual $600 rainy day investment.
   - savings account at 1.5% APR
   - certificate of deposit at 4.5% APR
   - stock investment at 10% APR

For each option, what is the amount of money that Kara earns if she invests for 3 years? For 10 years? For 30 years? Make a spreadsheet like the one below for each investment period.

2. Research the APR for savings and CD accounts at a local bank. Find the fees charged on minimum balances in various types of accounts. Which type of account is the best for investing a small amount? A large amount?

Comparing Payment Options

Make a spreadsheet showing the payment options for each situation below.

3. The Peña family finances a car for $8000. A car dealership offers a 7.9% APR loan for 4 years. Each month, the dealership charges interest on the balance of the loan. What is the monthly payment rounded to the nearest cent?
4. The Peñas need $32,000 in financing to buy their dream car.
   a. What is the family's monthly payment rounded to the nearest cent?
   b. How does this monthly payment compare to the monthly payment in Exercise 3?
5. The Peñas get a new finance offer for their dream car, a 5-year loan at 8.9% APR.
   a. How much does this reduce the family's monthly payment?
   b. How do the car payments in Exercises 4 and 5 compare?

	A	B	C	D	E
1	Year	Starting Balance	Interest	Deposit	Ending Balance
2	1	0	0	600	600
3					
4					

Answers

Chapter Project

1. $1827.14, $1882.22, $1986.00; $6421.63, $7372.93, $9562.45; $22,523.21, $36,604.24, $98,696.41
2. Check students' work.
3. $194.93
4. **a.** $779.71
   b. four times as great
5. **a.** $29.25
   b. $194.93 for 48 months = $9356.64; 165.68 for 60 months = $9940.80

Would a $500 discount at 6.4% interest for 4 years be even better?

6. The car dealership makes its final offers to the Peñas, a $1000 discount and 7.9% interest for 4 years, or no discount and 4.9% interest for 4 years. Which of these two offers should they take?

7. Nancy buys appliances for $1300 at Adequate Buy. She has two payment options. She can use a credit card that charges 10% APR, or she can use an Adequate Buy credit card that charges no interest for a year. Then she pays 15% APR after the first year.

   Which card should Nancy use if she plans to make monthly payments of $110? Of $40?

8. The United States government offers college student loans that typically carry lower interest rates than car loans. Shannon graduates with a $12,000 student loan. The interest rate on her loan is 5% APR.
   - **a.** Find Shannon's monthly payment rounded to the nearest cent.
   - **b.** What percentage of Shannon's first payment is interest?
   - **c.** Mark has a student loan of $18,000 at 5% APR. Estimate his monthly payment. Then find his monthly payment rounded to the nearest cent.
   - **d.** If Shannon's loan were a bank loan at 8% APR, how much more would her total payments be?

9. **Take It Further** Julie and Ben owe $215,354 on their existing mortgage. They have already paid for 3 years at 7% APR. A bank advertises a new 30-year loan at 6.5% APR. The bank charges $2500 in fees to issue the loan and adds the fee to the loan amount.
   - **a.** What is the monthly payment on Julie and Ben's existing mortgage?
   - **b.** If they choose to take a new loan, what will be the monthly payment?
   - **c.** How much money will they save, if any, by taking a new loan?
   - **d.** Experiment using a different bank rate or loan amount. Find how the monthly payment changes.

Go Online PHSchool.com

For more information on spreadsheets, go to **Web Code:** bdd-0561

EXERCISE 6 This might be a surprising result for students. They are likely to assume that the lower interest rate is the better choice.

EXERCISE 7 Make sure students remember that the Adequate Buy credit card interest rate changes after 12 months. They will need to reflect this change in their spreadsheets.

TAKE IT FURTHER Ask students to find the cutoff point where the 10% APR card becomes the better choice.

EXERCISE 9 Students may stumble when trying to figure out what changes to make to compare one loan to the other. The change in time frame from 27 years to 30 years may be especially challenging. This is the key to recognizing why refinancing may not be such a great deal. One useful extension is to look at a 6% APR bank offer, which actually does cost less over the life of the loan. The monthly payment would be $1306.14 and the total payments would be $470,210.40.

6. the first offer; lower monthly payments

7. Adequate Buy; Adequate Buy

8. **a.** $127.28
   b. 39.3%
   c. Estimates may vary. Sample: 1.5 times Shannon's payment, or about $190. monthly payment = $127.28.
   d. $2197.20

9. **a.** $1481.24
   b. $1276.99
   c. The new loan will cost nearly $16,000 more.
   d. Check students' work.

Lesson Overview

GOALS

- Find the monthly payment on a loan, given the interest, rate, term, and amount you borrow.
- Understand how the monthly payment on a loan changes with the amount you borrow.
- Build a recursively defined function that agrees with a table.

Students recursively define a function that determines the balance owed at the end of the nth month of a loan. They see how to make more efficient the calculations involved in computing the function values. They see the effect of inefficient calculations on a recursive function. Through the exercises, students examine the interplay between all of the variables in this situation: loan amount, term of the loan, interest rate, and monthly payment.

CHECK YOUR UNDERSTANDING
- Core: 1, 2, 3, 4
- Optional: 5

MATERIALS
- CAS (recommended)
- graphing calculators
- spreadsheet software

HOMEWORK
- Core: 6, 7, 8
- Optional: 9, 10

Launch

You might want to begin by having students discuss what they know about monthly payments, compound interest, and similar topics. Then work through For You to Do problems to look at loan balances for a 0% loan.

Explore

For You to Do

Students can define $b(n)$ on their CAS and calculate the balances $b(1)$, $b(2)$, $b(6)$, $b(12)$, and $b(36)$.

1.11 Monthly Payments

Suppose you want to buy a car. You can put $1000 down and pay $250 per month. The interest rate is 5%, and the dealer wants the loan paid off in three years. How much can you afford to spend on the car?

A recursive approach lets you experiment using your function modeling language. Can you afford a $10,000 car? Well, after you pay $1000 down, you can borrow $9000. At the end of 36 months, you want the balance to be 0.

Begin with a simpler model, one that would hold in an ideal world where you did not have to pay any interest.

- At the end of Month 0, you owe $9000 (that is how much you borrowed, and Month 0 is when the loan starts).
- For any month after Month 0, you owe $250 less than the month before (because you paid $250).

Every now and then, car dealers offer a 0% interest rate.

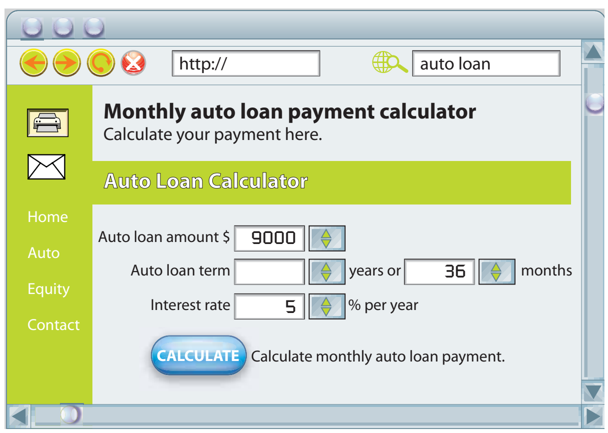
If $b(n)$ is the balance in dollars at the end of the nth month, then

$$b(n) = \begin{cases} 9000 & \text{if } n = 0 \\ b(n-1) - 250 & \text{if } n > 0 \end{cases}$$

For You to Do

For Problems 1–4, find how much you owe after each time span.

1. 1 month
2. 2 months
3. 6 months
4. 1 year
5. Can you pay off your car in 36 months? If not, how much do you still owe?



Go Online PHSchool.com

For a project with monthly payments, go to **Web Code:** bfe-9031

Answers

For You to Do

1. $8750
2. $8500
3. $7500
4. $6000
5. Yes; after 36 months you will have paid the entire $9000 loan.

For Discussion

6. Find a closed form for the function b on the previous page.

What happens when you take the 5% interest into account?

- Month 0 is when the loan starts. At the end of Month 0, you owe \$9000. That amount is how much you borrowed.
- For any month after Month 0, you owe the balance from the previous month, plus the interest on that balance, minus \$250.

The interest for one month is one twelfth the interest for a whole year. The interest for a year is 5%, so the interest for a month is $\frac{0.05}{12}$ times the balance for that month.

You can now refine the definition of b.

$$b(n) = \begin{cases} \$9000 & \text{if } n = 0 \\ b(n-1) + \frac{0.05}{12} \cdot b(n-1) - \$250 & \text{if } n > 0 \end{cases}$$

At the end of the first month you owe

- \$9000 (what you owed at the beginning of the month)
- plus $\frac{0.05}{12} \cdot \$9000$ (which is \$45.83)
- minus \$250 (your monthly payment)

$$\$9000 + \left(\frac{0.05}{12} \cdot \$9000\right) - \$250 = \$8787.50$$

Banks use the following rule: What you owe at the end of the month is what you owed at the start of the month, plus $\frac{1}{12}$ of the yearly interest on that amount, minus the monthly payment.

For You to Do

7. What do you owe at the end of Month 2?

8. What do you owe at the end of Month 3?

The Memory Error

When you model the new recursive definition for b, your system can quickly run out of memory. Why?

Your calculator may return a message such as "Recursion too deep."

How does a computer find $b(4)$ using this definition? You tell it that $n = 4$, and, as it scans the definition, it notes what it needs to do. It sees two places where it will need to compute $b(3)$.

$$b(4) = \boxed{b(3)} + \frac{0.05}{12} \cdot \boxed{b(3)} - 250$$

$$\boxed{b(2)} + \frac{0.05}{12} \cdot \boxed{b(2)} - 250 \qquad \boxed{b(2)} + \frac{0.05}{12} \cdot \boxed{b(2)} - 250$$

$$\boxed{?} \quad \boxed{?} \quad \boxed{?} \quad \boxed{?}$$

In each of these 2 computations of $b(3)$, the computer needs to compute $b(2)$ twice, for a total of 4 times. Continuing, it computes $b(1)$ 8 times, and

For Discussion

Students should recognize that this recursive definition can be defined by a closed-form linear function since you are adding a constant, −250.

For You to Do

Ask students to calculate the interest paid each month. They often do not realize that more of your payment goes toward interest at the beginning of the loan than at the end of the loan. Discuss why this is true.

The Memory Error

Let students experiment with different values of n. The calculation becomes noticeably slower as n gets larger. For very large values of n there is an error message that may read "Error: Resource Exhaustion."

For Discussion

6. $b(n) = 9000 - 250n$

For You to Do

7. \$8574.11

8. \$8359.84

For You to Do

Encourage students to build a spreadsheet on their CAS to answer Problem 10.

Wrap Up

Have students work on the core exercises at the end of class. For Exercise 3, ask students to determine the total amount of interest paid for each amount of time. Could this influence their choice of loans if they were not concerned with the amount of the monthly payment?

Assessment Resources

Lesson Quiz 1.11

1. Suppose you take a \$20,000 loan at 4% interest with a \$350 monthly payment. How much do you owe at the end of one year?
2. If you can only afford to pay \$330 per month, and you must pay your loan off in 48 months at 5% interest, how much money can you borrow?
3. What does the interest rate have to be so that you can afford a \$21,000 car with \$3000 down and payments of \$405 per month for 48 months?
4. A car dealer has an ad each week that offers two deals on its cars.
   - You can get a \$2500 rebate on the list price of the car. You then pay off the rest in 48 months at 7.9% interest.
   - Instead of the rebate, you can get a low 0.9% interest rate and then pay off the full list price of the car in 48 months.

   Suppose you want to buy a car with a list price of \$33,000. Is it better to take the rebate or the low interest rate deal?

the base case $b(0)$ 16 times. Then it feeds this information back through three layers of computation to do 1 last calculation to get $b(4)$.

Adding the number of evaluations of b, you have

$$1 + 2 + 4 + 8 + 16 = 31 = 2^{(4+1)} - 1$$

That is how many computations the computer needs to compute $b(4)$. In general, calculating $b(n)$ requires tracking $2^{(n+1)} - 1$ evaluations of b. No wonder your calculator or computer runs into trouble quickly.

But where your calculator sees two calculations to be done separately, a person would notice that they are the same—both $b(n-1)$. You can figure it out once and then plug it in both places.

$$b(4) = \boxed{b(3)} + \frac{0.05}{12} \cdot \boxed{b(3)} - 250$$

$$\boxed{b(2) + \frac{0.05}{12} \cdot b(2) - 250}$$

You can force the computer to do the calculation the smart way as well. If you take the expression $b(n-1) + \frac{0.05}{12} \cdot b(n-1)$ and factor out $b(n-1)$, you get the following expression.

$$\begin{aligned} b(n-1) + \frac{0.05}{12} \cdot b(n-1) &= 1 \cdot b(n-1) + \frac{0.05}{12} \cdot b(n-1) \\ &= b(n-1)\left(1 + \frac{0.05}{12}\right) \end{aligned}$$

Now the computer only needs to track $n + 1$ evaluations of b in order to find $b(n)$. This change makes the computation manageable.

Habits of Mind

Estimate. Estimating shows that $2^{10} > 1000$ and $2^{20} > 1{,}000{,}000$.

Smart calculation does not guarantee you will never get the memory error, just that you will not get it as soon!

For You to Do

9. Build a model in your function-modeling language for the function b below.

$$b(n) = \begin{cases} 9000 & \text{if } n = 0 \\ \left(1 + \frac{0.05}{12}\right) + b(n-1) - 250 & \text{if } n > 0 \end{cases}$$

10. How much do you owe after 6 months? How long does it take you to pay the loan down to \$8000?

You can add extra inputs that will let you change the monthly payment, the amount of the loan, or the interest rate on the loan. See the TI-Nspire Handbook, p. 778.

Answers

For You to Do

9. Check students' work.

10. \$7711.64; the balance is under \$8000 after 5 months.

Exercises Practicing Habits of Mind

Check Your Understanding

For Exercises 1–3, suppose you take a \$9000 loan at 5% interest with a \$250 monthly payment.

1. How much do you owe at the end of one year?
2. Can you pay off the loan in 36 months?
3. What monthly payment will let you pay off the loan in each amount of time?
   a. 36 months **b.** 39 months **c.** 48 months
4. If you can only afford to pay \$250 per month, and you must pay your loan off in 36 months at 5% interest, how much money can you borrow?
5. What does the interest rate have to be so that you can afford a \$12,000 car with \$1000 down and payments of \$310 per month for 36 months?

On Your Own

6. Suppose you want to pay off a car loan in 36 months. Pick an interest rate and keep it constant. Investigate how the monthly payment changes with the cost of the car.

   a. Make a table like the one below and complete it.

Cost of Car (thousands of dollars)	Monthly Payment
10	■
11	■
12	■
13	■
14	■
15	■
16	■
⋮	⋮

   b. Find either a closed-form or a recursive definition for a function that agrees with your table. Build a model of your function.

   c. Use your model to find the monthly payment on a \$26,000 car.

Make sure you state what interest rate you are using. You can find a rate in a newspaper car ad.

Go Online PHSchool.com

For additional practice, go to **Web Code:** bfa-0111

If this function works for all inputs, you can use it to calculate the monthly payment in terms of the cost of the car.

Exercises

1. \$6390.74
2. no; \$764.92 remains to be paid
3. **a.** \$269.74
   b. \$250.51
   c. \$207.26
4. \$8341.43
5. about 0.9%
6. **a.** See back of book.
   b. If A = amount borrowed, P = amount of each payment, m is the monthly interest rate $\left(m = \frac{r}{12}\right)$, then here are two closed form equations:
   $$P = \frac{A \cdot m(1+m)^{36}}{(1+m)^{36} - 1}$$ and
   $$A = \frac{P((1+m)^{36} - 1)}{m(1+m)^{36}}.$$
   c. Answers may vary. Sample: At 7% the monthly payment would be \$798.50.

Exercises

HOMEWORK
- Core: 6, 7, 8
- Optional: 9, 10

Check Your Understanding

EXERCISE 2 If $b(36) = 0$ the loan is paid off. If $b(36) > 0$ the loan is not paid off. If $b(36) < 0$, the loan is overpaid.

EXERCISE 3 This is a good exercise to add an extra input:

$$b(n, p) = \begin{cases} 9000, & n = 0 \\ \left(1 + \frac{0.05}{12}\right) \cdot b(n-1, p) - p, & n > 0 \end{cases}$$

Then you can use your CAS. Evaluate solve$(b(36, p) = 0, p)$ to get the correct payment.

On Your Own

EXERCISE 6 You can add a third input for the cost of the car:

$$b(n, p, c) = \begin{cases} c, & n = 0 \\ \left(1 + \frac{0.05}{12}\right) \cdot b(n-1, p, c) - p, & n > 0 \end{cases}$$

Use solve$(b(36, p, 10000) = 0, p)$ to find the first entry in the table. Then change the cost of the car to find other entries. Or, you could build a spreadsheet on your CAS.

EXERCISE 7 In Exercise 6 you found a closed-form definition to find the monthly payment. If you multiply the monthly payment by 36, you will get the actual cost of the car. You want to build a similar table for the low interest rate and compare the actual costs.

Maintain Your Skills

EXERCISE 9 Compare the answers to powers of 2. Each sum is one less than a power of 2.

Additional Resources

PRINT RESOURCES
- Solution Manual
- Practice Workbook
- Assessment Resources
- Teaching Resources

TECHNOLOGY
- Interactive Textbook
- TeacherExpress CD-ROM
- ExamView CD-ROM
- PHSchool.com

Additional Practice

For Exercises 1–3, suppose you take a $10,000 loan at 6% interest with a $275 monthly payment.

1. How much do you owe at the end of one year?
2. Can you pay off the loan in 36 months?
3. What monthly payment will let you pay off the loan in each amount of time?
   a. 36 months **b.** 39 months **c.** 48 months
4. If you can only afford to pay $250 per month, and you must pay your loan off in 48 months at 6% interest, how much money can you borrow?
5. What does the interest rate have to be so that you can afford a $15,000 car with $1000 down and payments of $300 per month for 48 months?
6. Suppose you want to pay off a car loan in 48 months. Pick an interest rate and keep it constant. Investigate how the monthly payment changes with the cost of the car.
   a. Complete the table.
   b. Find either a closed-form or a recursive definition for a function that agrees with your table. Build a model of your function.
   c. Use your model to find the monthly payment on a $25,000 car.

Cost of Car (thousands of dollars)	Monthly Payment
15	
16	
17	
18	
19	
20	
⋮	⋮

7. A car dealer has an ad each week that offers two deals on its cars.
   - You can get a $2500 rebate on the list price of the car. You then pay off the rest in 48 months at 6.9% interest.
   - Instead of the rebate, you can get a low 1.9% interest rate and then pay off the full list price of the car in 48 months.

   Cars on the lot sell at prices between $15,000 and $35,000.
   a. For which cars is it better to take the rebate?
   b. For which cars is it better to take the low interest rate?
   c. For what price are the deals the same?

Practice: For Lesson 1.11, assign Exercises 1–7.

7. A local car dealer has an ad each week that offers two deals on its cars.
- You can get a $2000 rebate on the list price of the car. You then pay off the rest in 36 months at 5% interest.
- Instead of the rebate, you can get a low 0.9% interest rate and then pay off the full list price of the car in 36 months.

Cars on the lot sell at prices between $20,000 and $40,000. For which cars is it better to take the rebate? For which cars is it better to take the low interest rate? For what price are the deals the same?

8. **Standardized Test Prep** Suppose you take a $10,000 loan at 5% interest. How much do you owe after three monthly payments of $500?

A. $1500.00 **B.** $8500.00 **C.** $8619.25 **D.** $8925.00

Habits of Mind

Organize what you know. Make a table for prices between $20,000 and $40,000 in increments of $1000. Figure out the monthly payment for each price. You can use your function-modeling language and a spreadsheet.

Maintain Your Skills

9. Evaluate each sum.

a. $1 + 2$

b. $1 + 2 + 2^2$

c. $1 + 2 + 2^2 + 2^3$

d. $1 + 2 + 2^2 + 2^3 + 2^4$

e. $1 + 2 + 2^2 + 2^3 + 2^4 + 2^5$

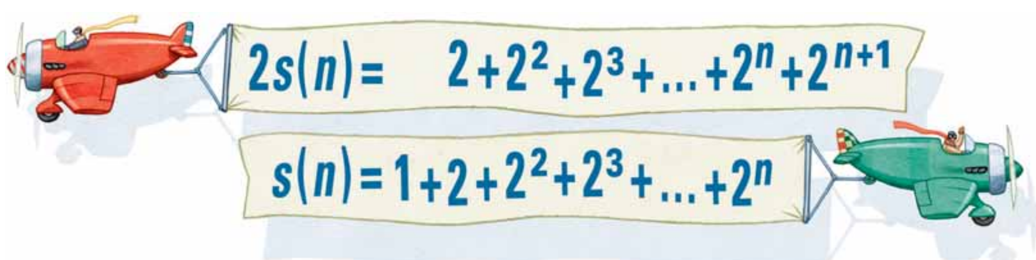
f. $1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + \cdots + 2^9$

10. Use the description of s below.

$$s(n) = 1 + 2 + 2^2 + 2^3 + \cdots + 2^n$$

a. Find a recursive definition for s.

b. Find a closed form for s.



Answers

7. Check students' tables. Answers may vary. Sample: For cars that cost less than $33,200, you would pay less over the life of the loan if you chose the rebate deal. For cars that cost more than $33,200, you would pay less if you chose the low interest rate deal. The deals are the same if the car costs $33,200.

8. C

9. a. 3
b. 7
c. 15
d. 31
e. 63
f. 1023

10. a.
$$s(n) = \begin{cases} 1 & \text{if } n = 0 \\ s(n-1) + 2^n & \text{if } n > 0 \end{cases}$$

b. $s(n) = 2^{n+1} - 1$

Project

This project gives students a closer look at the function that determines the balance on a loan given the original loan amount, the interest rate, the monthly payment, and the term of the loan (the number of months it takes to pay it off).

Although students have already worked fairly extensively with this monthly payment situation, they have not as yet done a general mathematical investigation of this function and how all of the variables interact. By keeping two of the variables constant, they will be able to see how the remaining quantities affect each other. Their goal is to express this relationship as a closed-form and/or recursive function.

If you like, you can have students work with a partner. The project becomes more personal if the students look in newspapers or online to find a car they would be interested in buying and use that in their project.

You might consider asking for preliminary drafts so that you can give students early feedback and make sure that they are progressing. Although evaluating these drafts creates more work for you, giving feedback on early drafts will lead to better final write-ups.

After students have completed the project, you might ask several students to present their results for each of the exercises. Students should be aware that no matter what constant values were chosen, the type of relationship for the remaining variables is unchanged. For example, in the first exercise no matter what interest rate was chosen for a 36-month loan, the relationship between the monthly payment and the cost of the car will be linear. Point out the type of relationship found in each of the other exercises.

EXERCISE 1 Students should pick a reasonable interest rate. They can find this information online, at their local bank, or in ads in the newspaper. They should complete the table using their monthly payment function and enter it as a spreadsheet on their calculator or computer. Graphing the data will give them a good idea what kind of function they are looking for. They can find that function on their own, or use the calculator to find the regression equation.

Project: Using Mathematical Habits

More On Monthly Payments

You need to consider several variables when you use a loan to buy a car:

- the cost of the car
- the interest rate
- the monthly payment
- the number of months that you will take to pay off the loan

The general function b below relates these variables. It gives the balance at the end of n months on a loan amount of c dollars at an interest rate of i% (represented by a decimal) with a monthly payment of m dollars.

$$b(n, c, i, m) = \begin{cases} c & \text{if } n = 0 \\ (1 + i)b(n - 1, c, i, m) - m) & \text{if } n > 0 \end{cases}$$

In this project, you will look at the effects of two of the variables on the rest.

- If i and n are constant, how does c depend on m?
- If c and n are constant, how does m depend on i?
- If i and c are constant, how does m depend on n?

If you finance $9000 at 5% interest for 36 months to buy this car, you will pay $9710.57 in all.

Cost and Monthly Payment

1. Pick an interest rate and keep it constant. Suppose you want to pay off a car loan in 36 months. Investigate how much you can spend on a car for a given monthly payment.

a. Make a table like the one below.

Monthly Payment	Cost of Car
200	■
210	■
220	■
230	■
240	■
250	■
260	■
270	■
280	■
⋮	⋮

b. Find either a closed-form or a recursive definition that lets you calculate the cost of the car in terms of the monthly payment. Build a model for your function.

c. Use your model from part (b) to find the cost of a car you can afford with a $360 monthly payment. Check your result with the original model for b.

Answers

Project

1. Check students' work.

Interest Rate and Monthly Payment

2. Investigate the effect of interest on the monthly payment. Pick a car price and keep it constant. Suppose you want to pay off a car in 36 months. The interest rate might range from 0% to 15%.

a. Make a table like the one below.

Interest Rate (%)	Monthly Payment
0	■
1	■
2	■
3	■
4	■
5	■
6	■
7	■
8	■
⋮	⋮

b. Find the monthly payment on the car when the interest rate is 13%.

c. **Take It Further** Find either a closed-form or a recursive definition that lets you calculate the monthly payment in terms of the interest rate. Express the interest rate as a decimal or a percent. Build a model for your function.

d. **Take It Further** Use the model you wrote in part (c) to find the monthly payment on your car when the interest rate is 13%. Check your result with the original model for *b*.

Length of Loan and Monthly Payment

3. Investigate the effect of the length of the loan on the monthly payment. Pick a car price and keep it constant. Find the monthly payment on the car as the number of months varies.

a. Make a table like the one below.

Term (months)	Monthly Payment
24	■
28	■
32	■
36	■
40	■
44	■
48	■
52	■
56	■
⋮	⋮

b. Find the monthly payment on the car when the term of the loan is five years.

c. **Take It Further** Find either a closed-form or a recursive definition that lets you calculate the monthly payment in terms of the length of the loan. Build a model for your function.

d. **Take It Further** Use the model you wrote in part (c) to find the monthly payment on the car when the length of the loan is five years. Check your result with the original model for *b*.

2–3. Check students' work.

EXERCISE 2 Students can search for a car that they would like to buy online or in the newspaper. They can fill in the table and enter the data in a spreadsheet. Have them graph the data and determine the type of function they are looking for. Ask: Does the monthly payment increase by some constant amount as the interest rate increases?

EXERCISE 3 When students graph the data for this exercise, they will see that it is not linear. They should use their calculator to find different regression equations and determine whether quadratic, power, or exponential gives the best fit.

This is a good exercise to talk about the total interest paid. Ask students: Do you pay the same total amount when the term is 24 months as when the term is 48 months? You might want to talk about being an informed consumer.

Lesson Overview

GOAL

- Find general classes of functions that fit a recurrence.

This lesson reminds students about situations they encountered where recurrences and closed forms are both involved. The main goal is for students to identify that linear, polynomial, and exponential rules can all emerge from recurrences.

CHECK YOUR UNDERSTANDING
- Core: 1, 3a–b, 4, 5
- Optional: 2, 6, 7
- Extension: 3c

MATERIALS
- CAS (recommended)
- graphing calculators
- Blackline Master 5.12

HOMEWORK
- Core: 8, 11a–b, 12, 14, 15
- Optional: 9, 10
- Extension: 11c, 13

VOCABULARY
- functional equation
- recurrence
- two-term recurrence

Launch

Review Problems 1 and 2 from the Getting Started lesson if you have not already done so. Then introduce some examples of recurrences like the ones presented in the Facts and Notation section.

5.12 Recurrences

In your studies, you have encountered recursive definitions for functions, sequences, and series. Examples include Problems 2 and 4 and Exercise 11 from Lesson 5.11. Here are some others.

- The function that gives the sum of the first n squares is

$$S(n) = \begin{cases} 0 & \text{if } n = 0 \\ S(n-1) + n^2 & \text{if } n > 0 \end{cases}$$

- The function that gives the monthly balance on a loan of \$10,000 at 5% APR with a monthly payment of \$520 is

$$b(n) = \begin{cases} 10000 & \text{if } n = 0 \\ \left(1 + \frac{0.05}{12}\right) \cdot b(n-1) - 520 & \text{if } n > 0 \end{cases}$$

- The function that gives the Fibonacci numbers is

$$F(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F(n-1) + F(n-2) & \text{if } n > 1 \end{cases}$$

The goal of this investigation is to develop some general purpose tools for finding closed-from equivalents for such definitions, and for proving that the closed-form functions actually do agree with the recursively defined functions.

Facts and Notation

- A **functional equation** tells how various outputs of a function are related. For example,

$$f(2x) = (f(x))^2$$

is a functional equation. Not every function satisfies this equation. But $r(x) = 3^x$ does, because

$$\begin{aligned} r(2x) &= 3x^2 \\ &= (3^x)^2 \\ &= (r(x))^2 \end{aligned}$$

- A **recurrence** is a special kind of functional equation that tells how the output of a function at integer n is related to the outputs at integers less than n. Examples include the recurrences that are satisfied by

arithmetic sequences. For example,

$$f(n) = f(n-1) + 3$$

Habits of Mind

Develop your understanding. A functional equation is an equation whose "variable" is a function. What are some other functions that satisfy this functional equation?

More complicated recurrences include the functional equations satisfied by binomial coefficients, and the function that gives the balance on a loan at the end of each month.

Geometric sequences. For example,

$$f(n) = 3f(n-1)$$

Fibonacci numbers.

$$f(n) = f(n-1) + f(n-2)$$

A recurrence like the last one, which defines the output at n in terms of the outputs at $n-2$ and $n-1$, is called a two-term recurrence.

As you saw in Lesson 5.1, a recurrence is not enough to define a function. You also need one or more base cases. For example, to complete the definition of an arithmetic sequence, you have to give the value for $f(0)$. See Problem 3 of Lesson 5.11 for an example of how different base cases produce different functions that satisfy the same recurrence.

The problem for this investigation is as follows.

Problem

a. Characterize all functions that satisfy a given recurrence.

b. Given base cases, find (if possible) a closed-form definition that satisfies the recurrence.

Suppose you have a recursive function definition. Here are some useful methods for finding a closed-form equivalent.

- **Model the recursive definition in your FML.** This gives you a computational model, with which you can experiment. You can look at outputs, look for patterns in the outputs, tabulate the model, graph it, and so on.
- **Tabulate the function.** Search the table for patterns. Make a difference table. If the third differences are constant, there is a cubic fit. Look at the ratios. If the successive ratios are constant, there is an exponential fit. Look at how the length of the string of digits grows. Rapid growth may point to a closed form based on an exponential function.
- **Graph the function.** For a recursively defined function, the domain is probably positive integers, in which case you will have to use a scatter plot. The shape of the plot can reveal whether the function is linear, exponential, or polynomial. A graph that looks quadratic would lead you to try a quadratic fit.
- **Use what you know from previous chapters and courses.** For example, you can write a recurrence like

  $$f(n) = 3f(n-1) + 2$$

  as

  $$f(n) = \mathcal{A}_{(3,2)}(f(n-1))$$

  where $\mathcal{A}_{(a,b)}(x) = ax + b$. In *Algebra 2* you learned how to "unstack" such recurrences using the iteration formula

  $$\mathcal{A}^{(n)}_{(a,b)} = \mathcal{A}\left(a^n, b\left(\frac{a^n-1}{a-1}\right)\right)$$

Remember...

$\mathcal{A}^{(n)}_{(a,b)}$ is $\mathcal{A}_{(a,b)}$ applied n times.

Explore

Problem

Remind students that the recurrence can only give a general form, while the base cases determine the specific instance of that form. Typically, then, they use the base cases to solve an equation (or a system of equations) to determine the coefficients in the solution.

In-Class Experiment

Watch for students using a table or graph to help them answer these questions. Also, consider skipping some parts or emphasizing others.

For Discussion

PROBLEMS 6–9 Answers will vary. You might consider assigning only Problems 6 and 7 here.

Wrap Up

Review Exercise 3, or build a similar two-term recurrence such as

$$f(n) = 8f(n-1) - 15f(n-2)$$

A ratio table is most helpful to see that the ratio of consecutive terms approaches 5. Here a closed form is

$$f(n) = h \cdot 5^n + j \cdot 3^n$$

Assessment Resources

Lesson Quiz 5.12

1. Define the function p recursively as

$$p(x) = \begin{cases} 54 & \text{if } x = 0 \\ \frac{2}{3}p(x-1) & \text{if } x > 0 \end{cases}$$

a. Calculate $p(4)$.
b. Find a closed-form definition for a function that agrees with p.
c. Find the value of the sum $\sum_{k=0}^{\infty} p(k)$.

2. Define the function f recursively as

$$f(x) = 3f(x-1)$$

with $f(0) = 2$.
a. Find a closed-form definition for a function that agrees with f.
b. Find the value of the sum $\sum_{k=0}^{10} f(k)$.

A recurrence like

$$f(n) = f(n-1) + 5^n$$

unstacks to a geometric series. You know how to sum those.

In-Class Experiment

Use the methods above, or anything else you like, to find closed-form equivalents for the following functions.

1. $f(n) = \begin{cases} 3 & \text{if } n = 0 \\ f(n-1) + 7 & \text{if } n > 0 \end{cases}$

2. $f(n) = \begin{cases} 3 & \text{if } n = 0 \\ 7f(n-1) & \text{if } n > 0 \end{cases}$

3. $f(n) = \begin{cases} 3 & \text{if } n = 0 \\ 4f(n-1) - 6 & \text{if } n > 0 \end{cases}$

4. $f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 3 & \text{if } n = 1 \\ 5f(n-1) - 6f(n-2) & \text{if } n > 1 \end{cases}$

5. $f(n) = \begin{cases} 2 & \text{if } n = 0 \\ 5 & \text{if } n = 1 \\ 5f(n-1) - 6f(n-2) & \text{if } n > 1 \end{cases}$

For Discussion

What kind of recurrence relation will each type of function satisfy?

6. linear
7. exponential
8. quadratic
9. degree-n polynomial

Computers that Prove Theorems

Babbage's difference engine performed calculations to find specific numerical outputs of polynomial functions. Today, computers can produce induction proofs of general theorems, like the identity

$$\sum_{k=1}^{n} k^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$$

Even more surprisingly, computers can discover and prove new identities, ones which mathematicians had not previously suspected.

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For more information on computer-based proofs, go to **Web Code:** bge-8031

Answers

In-Class Experiment

1. $f(n) = 7n + 3$
2. $f(n) = 3 \cdot 7^n$
3. $f(n) = 4^n + 2$
4. $f(n) = 3^n$
5. $f(n) = 2^n + 3^n$

For Discussion

6. $f(n) = f(n-1) + k$
7. $f(n) = f(n-1) \cdot k$
8. $f(n) = f(n-1) + An + B$
9. $f(n) = f(n-1) + p(n)$, where $p(n)$ is a polynomial of degree $(n-1)$

Exercises *Practicing Habits of Mind*

Check Your Understanding

1. Define function f recursively as

$$f(n) = \begin{cases} K & \text{if } n = 0 \\ 3f(n-1) & \text{if } n > 0 \end{cases}$$

Suppose also that $f(5) = 100$.

a. Calculate $f(6)$ and $f(4)$.

b. Find the value of K.

c. Find a closed-form definition for a function that agrees with f.

2. Define function g recursively, by

$$g(n) = \begin{cases} 64 & \text{if } n = 0 \\ \frac{3}{4}g(n-1) & \text{if } n > 0 \end{cases}$$

a. Calculate $g(4)$.

b. Find a closed-form definition for a function that agrees with g.

c. Find the value of this sum.

$$\sum_{j=0}^{\infty} g(j)$$

d. Approximate the value of this sum to four decimal places.

$$\sum_{j=0}^{100} g(j)$$

3. a. Find the bases for two different exponential functions $h(n) = a^n$ and $j(n) = b^n$ that satisfy the recurrence

$$f(n) = 10f(n-1) - 24f(n-2)$$

b. For your functions h and j in part (a), define

$$k(n) = 5h(n) + 8j(n)$$

Show that k also satisfies the recurrence

$$f(n) = 10f(n-1) - 24f(n-2)$$

c. Take It Further Suppose a function r satisfies the recurrence given above, and $r(0) = 3$ and $r(1) = 8$. Find a closed-form definition for a function that agrees with r.

Habits of Mind

Generalize. There is nothing special about 5 and 8 here.

Exercises

HOMEWORK

- Core: 8, 11a–b, 12, 14, 15
- Optional: 9, 10
- Extension: 11c, 13

Check Your Understanding

EXERCISE 1 If you discuss this exercise, mention briefly that $f(n) = 100 \cdot 3^{n-5}$ is a good closed-form definition since the initial value was given as $f(5)$ rather than $f(0)$.

EXERCISE 2 Use this exercise to remind students how to find the sum of a geometric series, which is handy throughout this investigation.

EXERCISE 3 previews the process for a two-term recurrence, the focus of the next lesson. So, give them a chance with this one, but do not look for mastery.

Exercises

1. a. $f(6) = 300$; $f(4) = \frac{100}{3}$

b. $K = \frac{100}{243}$ or $\frac{100}{3^5}$

c. $f(n) = 100 \cdot 3^{n-5}$

2. a. $g(4) = 20.25$

b. $g(n) = 64 \cdot \left(\frac{3}{4}\right)^n$

c. 256

d. 256.0000

3. a. $h(n) = 4^n$; $j(n) = 6^n$

b.
$$\begin{aligned} 10k(n-1) - 24k(n-2) &= 10(5 \cdot 4^{n-1} + 8 \cdot 6^{n-1}) - 24(5 \cdot 4^{n-2} + 8 \cdot 6^{n-2}) \\ &= 50 \cdot 4^{n-1} + 80 \cdot 6^{n-1} - 120 \cdot 4^{n-2} - 192 \cdot 6^{n-2} \\ &= 50 \cdot 4^{n-1} + 80 \cdot 6^{n-1} - 30 \cdot 4 \cdot 4^{n-2} - 32 \cdot 6 \cdot 6^{n-2} \\ &= 50 \cdot 4^{n-1} + 80 \cdot 6^{n-1} - 30 \cdot 4^{n-1} - 32 \cdot 6^{n-1} \\ &= 20 \cdot 4^{n-1} + 48 \cdot 6^{n-1} \\ &= 5 \cdot 4 \cdot 4^{n-1} + 8 \cdot 6 \cdot 6^{n-1} \\ &= 5 \cdot 4^n + 8 \cdot 6^n \\ &= k(n) \end{aligned}$$

c. $r(n) = 5 \cdot 4^n - 2 \cdot 6^n$

EXERCISE 4 is suspiciously simple, but later exercises follow the same pattern with more complicated recurrences.

EXERCISE 5 This method, based on the fixed point, is one way to get closed-form function definitions that agree with recursive ones in the form $f(x) = Af(x-1) + B$, like those used for annuities and loan payments. This exercise serves as a preview to the material on annuities and loan payments later in this investigation.

EXERCISE 7 You can make copies of Blackline Master 5.12 for students to use with this exercise.

4. Suppose f is a function that satisfies the recurrence $f(n) = 2f(n-1)$.

a. Find a value b for $f(0)$ that makes f a constant function.

b. Judy starts with $f(0)$ a distance of 16 units away from b. That is, $|f(0) - b| = 16$. How far away from b is $f(1)$? $f(2)$?

5. Josue considers the more complicated recurrence

$$j(n) = 2j(n-1) + 3$$

a. What base case $j(0) = b$ makes j a constant function?

b. Josue starts with $j(0)$ a distance of 16 units away from b. How far away from b is $j(1)$? $j(2)$?

6. Consider function g from Exercise 2.

a. Find $\lim_{n\to\infty} g(n)$.

b. How far away from this limit is $g(0)$? $g(1)$? $g(2)$?

7. Consider the recurrence

$$r(x) = \frac{3}{4}r(x-1) + 10$$

a. Find a closed-form definition for a function that agrees with r if $r(0) = 40$.

b. Copy and complete the following table for the base case $r(0) = 104$.

n	$r(n)$	Δ	$\div$
0	104	■	■
1	■	■	■
2	■	■	■
3	■	■	■
4	■	■	■
5	■	■	■
6	■	■	■
7	■	■	
8	■		

The $\div$ column consists of the ratios of successive entries in the Δ column—the ratios of successive differences.

Answers

4. a. $b = 0$

b. $f(1)$ is 32 away from b and $f(2)$ is 64 away from b.

5. a. $b = -3$

b. $j(1)$ is 32 away from b and $j(2)$ is 64 away from b.

6. a. 0

b. $g(0)$ is 64 away from the limit, $g(1)$ is 48 away from the limit, $g(2)$ is 36 away from the limit.

7. a. $r(x) = 40$

b.

n	$r(n)$	Δ	$\div$
0	104	−16	0.75
1	88	−12	0.75
2	76	−9	0.75
3	67	−6.75	0.75
4	60.25	−5.0625	0.75
5	55.1875	−3.7969	0.75
6	51.3906	−2.8477	0.75
7	48.5430	−2.1357	
8	46.4072		

On Your Own

8. Establish the following identity.

$$(1 + r + r^2 + r^3 + \cdots + r^{n-1})(1 - r) = 1 - r^n$$

So, for any number $r \neq 1$,
$1 + r + r^2 + r^3 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r}$

9. **Write About It** The formula for the sum of an infinite geometric series when $|r| < 1$ is

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1 - r}$$

Why is $|r| < 1$ required? Why does this formula not work for $|r| \geq 1$?

10. Define function f recursively as

$$f(x) = 4f(x - 1)$$

with $f(0) = 3$.

a. Find a closed-form definition for a function that agrees with f.

b. Find the value of

$$\sum_{k=0}^{10} f(k)$$

11. a. Find the bases for two different exponential functions $h(n) = a^n$ and $j(n) = b^n$ that satisfy the recurrence

$$f(n) = 11f(n - 1) - 24f(n - 2)$$

b. For your functions h and j in part (a), define

$$k(n) = 7h(n) - 2j(n)$$

Show that k also satisfies the recurrence

$$f(n) = 11f(n - 1) - 24f(n - 2)$$

c. **Take It Further** Suppose a function r satisfies the recurrence given above and $r(0) = 5$ and $r(1) = 10$. Find a closed-form definition for a function that agrees with r.

Habits of Mind

Generalize. There is nothing special about 7 and -2 here.

12. Here is a recurrence.

$$f(n) = 5f(n - 1) - 6f(n - 2)$$

For each sequence below, decide whether or not the sequence could be consecutive outputs of a function that satisfies the recurrence.

a. 1, 2, 4, 8, 16, 32 **b.** 1, 3, 9, 27, 81, 243

c. 2, 5, 13, 35, 97, 275 **d.** 1, 4, 16, 64, 256, 1024

e. 0, 0, 0, 0, 0, 0

f. $a + b$, $2a + 3b$, $4a + 9b$, $8a + 27b$, $16a + 81b$, $32a + 243b$

g. 7, 13, 23, 37, 47, 13

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On Your Own

EXERCISE 12 Point out to students that the sequence in part (f) is a times the first plus b times the second. This is the general tactic that is used in the next lesson: find two exponential functions that satisfy the recursive definition, then show that you can solve any initial values by a times the first exponential function plus b times the second.

8. Expand the left side to $1 + r^2 + r^3 + \cdots + r^{n-1}$, then multiply by -1 to get $-1 - r^2 - r^3 - \ldots - r^{n-1}$. Adding together these terms leaves just $(1 - r^n)$ as the result of the expansion.

9. If $|r| > 1$, then the terms get infinitely larger, not smaller.

10. a. $f(n) = 3 \cdot 4^n$

b. 4,194,303

11. a. $h(n) = 3^n$, $j(n) = 8^n$

b.
$$\begin{aligned} 11k(n-1) - 24k(n-2) &= 11(7 \cdot 3^{n-1} - 2 \cdot 8^{n-1}) - 24(7 \cdot 3^{n-2} + 2 \cdot 8^{n-2}) \\ &= 77 \cdot 3^{n-1} - 22 \cdot 8^{n-1} - 168 \cdot 3^{n-2} + 48 \cdot 2^{n-2} \\ &= 77 \cdot 3^{n-1} - 22 \cdot 8^{n-1} - 56 \cdot 3 \cdot 3^{n-2} + 6 \cdot 8 \cdot 8^{n-2} \\ &= 77 \cdot 3^{n-1} - 22 \cdot 8^{n-1} - 56 \cdot 3^{n-1} - 6 \cdot 8^{n-1} \\ &= 21 \cdot 3^{n-1} - 16 \cdot 8^{n-1} \\ &= 7 \cdot 3 \cdot 3^{n-1} + 2 \cdot 8 \cdot 8^{n-1} \\ &= 7 \cdot 3^n + 2 \cdot 8^n \\ &= k(n) \end{aligned}$$

c. $r(n) = 6 \cdot 3^n - 8^n$

12. a. yes **b.** yes **c.** yes **d.** no
e. yes **f.** yes **g.** yes

Maintain Your Skills

EXERCISE 15 You can do this exercise efficiently by using the calculator and spreadsheet interaction on the nSpire. On one screen, define (and redefine) $t(n)$, while a second screen has an input-output and ratio table. Students may want to add extra columns such as the result of $t(n) - 3^n$ to find the other part of the closed form.

This exercise may take a while. You might consider only assigning one or two parts. The closed form in Part (c) is the simplest to identify, since it involves powers of 10. Still, not all students find the closed form here, and you should consider this exercise a chance for students to figure out the rules governing closed forms before the formality of the next lesson.

Additional Resources

PRINT RESOURCES
- Solution Manual
- Practice Workbook
- Assessment Resources
- Teaching Resources

TECHNOLOGY
- Interactive Textbook
- TeacherExpress CD-ROM
- ExamView CD-ROM
- PHSchool.com

Additional Practice

1. Define function f recursively as

$$f(n) = \begin{cases} c & \text{if } n = 0 \\ 2 \cdot f(n-1) & \text{if } n > 0 \end{cases}$$

Suppose also that $f(4) = 80$.
a. Calculate $f(3)$ and $f(5)$.
b. Find the value of c.
c. Find a closed-form definition for a function that agrees with f.

2. Define function g recursively as

$$g(x) = \begin{cases} 12 & \text{if } x = 0 \\ \frac{1}{3}g(x-1) & \text{if } x > 0 \end{cases}$$

a. Calculate $g(6)$.
b. Find a closed-form definition for a function that agrees with g.
c. Find the value of this sum.

$$\sum_{k=0}^{\infty} g(k)$$

3. Define f recursively as

$$f(x) = 5 \cdot f(x-1)$$

with $f(0) = 10$.
a. Find a closed-form definition for a function that agrees with f.
b. Find the value of

$$\sum_{k=0}^{10} f(k)$$

4. Here is a recurrence.

$$f(n) = 3f(n-1) - 2f(n-2)$$

For each sequence below, decide whether or not the sequence could be consecutive outputs of a function that satisfies the recurrence.
a. 0, 1, 3, 7, 15, 31
b. 0, a, $3a$, $4a$, $6a$, $10a$
c. 2, 3, 5, 7, 9, 11
d. 10, 15, 55, 135, 295, 615

Practice: For Lesson 5.12, assign Exercises 1–4.

13. **Take It Further** Find a closed-form definition for a function f that generates the sequence from Exercise 12g,

$$7, 13, 23, 37, 47, 13, \ldots$$

and continues to satisfy the recurrence

$$f(n) = 5f(n-1) - 6f(n-2)$$

14. **Standardized Test Prep** This is the top row of the difference table for a polynomial function of degree 4.

Input, n	Output, $f(n)$	Δ	Δ^2	Δ^3	Δ^4
0	z	r	m	c	1

Which of the following is the $n = 3$ row?

a.

3	$z + 3r$	$r + 3m$	$m + 3c$	$c + 3$	1

b.

3	$4z + 2r + m$	$4r + 2m + c$	$4m + 2c + 1$	$4c + 2$	1

c.

3	z^3	r^3	m^3	c^3	1

d.

3	$z + 3r + 3m + c$	$r + 3m + 3c + 1$	$m + 3c + 3$	$c + 3$	1

Maintain Your Skills

15. Given each recursively-defined function,
- Tabulate t using inputs from 0 to 7.
- Describe what is happening to the ratio of consecutive terms as the input grows.
- Find a closed-form definition for a function that agrees with t.

a. $t(n) = 5t(n-1) - 6t(n-2)$, with $t(0) = 2$ and $t(1) = 5$
b. $t(n) = 3t(n-1) + 10t(n-2)$, with $t(0) = 2$ and $t(1) = 3$
c. $t(n) = 13t(n-1) - 30t(n-2)$, with $t(0) = 2$ and $t(1) = 13$
d. $t(n) = -7t(n-1) - 12t(n-2)$, with $t(0) = 2$ and $t(1) = -7$
e. $t(n) = 7t(n-1) - 12t(n-2)$, with $t(0) = 2$ and $t(1) = 7$
f. $t(n) = 6t(n-1) + 40t(n-2)$, with $t(0) = 2$ and $t(1) = 6$

Habits of Mind

Look for a pattern. All the closed-form definitions will be similar. If you can find one, it will be easier to find the others.

Answers

13. $f(n) = 8 \cdot 2^n - 3^n$ **14.** D

15. a.

n	$t(n)$
0	2
1	5
2	13
3	35
4	97
5	275
6	793
7	2315

The ratio of consecutive terms is approaching 3.
$f(n) = 2^n + 3^n$

5.13 $f(n) = Af(n-1) + Bf(n-2)$

What can you say about a function that satisfies this recurrence?

$$f(n) = 7f(n-1) - 10f(n-2)$$

For the sake of illustration, try to find a closed-form definition for a function that agrees with g, which is f with base cases specified as shown:

$$g(n) = \begin{cases} 4 & \text{if } n = 0 \\ 11 & \text{if } n = 1 \\ 7g(n-1) - 10g(n-2) & \text{if } n > 1 \end{cases}$$

Start by building a model for g and tabulating it:

n	$g(n)$
0	4
1	11
2	37
3	149
4	673
5	3221
6	15,817
7	78,509
8	391,393
9	1,954,661
10	9,768,697
11	48,834,269
12	244,152,913
13	1,220,727,701
14	6,103,564,777
15	30,517,676,429
16	152,588,087,233
17	762,939,846,341
18	3,814,698,052,057
19	19,073,487,900,989
20	95,367,434,786,353

Habits of Mind

Use a model. Model g in your FML.

Habits of Mind

Visualize. Graph a scatter plot of $g(n)$ against n as well.

Lesson Overview

GOAL

- Relate a solution for a two-term recurrence to a quadratic polynomial.

This lesson targets two-term recurrences, and gives reasons why their closed forms are the sum of exponentials. Concentrate on the characteristic polynomial, whose roots are the bases of the exponentials.

CHECK YOUR UNDERSTANDING
- Core: 1a–e, 2, 3, 7
- Optional: 4, 5, 6
- Extension: 1f

MATERIALS
- graphing calculators
- Blackline Master 5.13

HOMEWORK
- Core: 8, 9, 10, 12, 16, 17
- Optional: 11
- Extension: 13, 14, 15

Launch

Begin today's lesson by reviewing Exercise 15 from the previous lesson, and see if students came up with any closed forms or generalizations.

You may want to use Blackline Master 5.13 as an overhead as you walk the students through the tabulation of function g.

b.

n	$t(n)$
0	2
1	3
2	29
3	117
4	641
5	3093
6	15,689
7	77,997

The ratio of consecutive terms is approaching 5. $f(n) = 5^n + (-2)^n$

c.

n	$t(n)$
0	2
1	13
2	109
3	1027
4	10,081
5	100,243
6	1,000,729
7	10,002,187

The ratio of consecutive terms is approaching 10. $f(n) = 10^n + 3^n$

d–f. See back of book.

The length of the digit strings suggests exponential growth. To test this, add a column with the ratios of consecutive terms.

n	$g(n)$	$\frac{g(n+1)}{g(n)}$
0	4	2.75
1	11	3.36364
2	37	4.02703
3	149	4.51678
4	673	4.78603
5	3221	4.91059
6	15,817	4.96358
7	78,509	4.98533
8	391,393	4.99411
9	1,954,661	4.99764
10	9,768,697	4.99906
11	48,834,269	4.99962
12	244,152,913	4.99985
13	1,220,727,701	4.99994
14	6,103,564,777	4.99998
15	30,517,676,429	4.99999
16	152,588,087,233	5.00000
17	762,939,846,341	5.00000
18	3,814,698,052,057	5.00000
19	19,073,487,900,989	5.00000
20	95,367,434,786,353	

Depending on your system, you may get slightly different numbers for the ratios. The constant 5 is an approximation—the actual ratio is very close to 5.

It looks as if the growth is "almost exponential." In fact, it looks as if the function $n \mapsto 5^n$ is close to a solution. What is going on?

Tabulate $n \mapsto 5^n$ next to g.

The answer is hinted at in Exercise 3 in Lesson 5.12. There are exponential functions of the form $n \mapsto a^n$ that satisfy the recurrence

$$g(n) = 7g(n-1) - 10g(n-2)$$

but they do not satisfy the base cases

$$g(0) = 4 \text{ and } g(1) = 11$$

But all is not lost. By transforming the exponential solutions a little, you can get a closed-form definition for a function that agrees with g. Here is how:

Suppose $n \mapsto a^n$ satisfies the recurrence

$$g(n) = 7g(n-1) - 10g(n-2)$$

Then $a^n = 7a^{n-1} - 10a^{n-2}$ for all integers $n \geq 2$.

Since the base of an exponential function cannot be 0, you can divide both sides by a^{n-2}. You get

$$a^2 = 7a - 10$$

This is a quadratic equation. So, if $n \mapsto a^n$ satisfies the recurrence, a must be a root of the quadratic equation

$$a^2 - 7a + 10 = 0$$

The polynomial $x^2 - 7x + 10$ is called the *characteristic polynomial* for the recurrence $g(n)$.

The roots of this equation are 2 and 5 (there is the 5 again). Just to be sure, check it out:

For You to Do

Show the following for all integers $n \geq 2$.

1. $2^n = 7 \cdot 2^{n-1} - 10 \cdot 2^{n-2}$
2. $5^n = 7 \cdot 5^{n-1} - 10 \cdot 5^{n-2}$

But not only does $n \mapsto 2^n$ satisfy the recurrence, so does the more general exponential function $n \mapsto k \cdot 2^n$, for any number k. To see this, multiply both sides of the equation

$$2^n = 7 \cdot 2^{n-1} - 10 \cdot 2^{n-2}$$

by k. If you distribute the k on the right side, you get

$$k \cdot 2^n = 7 \cdot k \cdot 2^{n-1} - 10 \cdot k \cdot 2^{n-2}$$

So, $n \mapsto k \cdot 2^n$ satisfies the recurrence

$$g(n) = 7 \cdot g(n-1) - 10 \cdot g(n-2)$$

Similarly, any function of the form $n \mapsto j \cdot 5^n$ for some constant j will also satisfy the recurrence.

Minds in Action

episode 16

Tony and Sasha are thinking about the solutions to the recurrence.

Tony We now have lots of functions that satisfy the recurrence. Things like $n \mapsto 3 \cdot 2^n$ or even $n \mapsto \pi \cdot 5^n$.

Sasha But none of them satisfy the base cases. They can't, because, for example, $j \cdot 5^n$ is j when $n = 0$, so j would have to be 4 if the first base case is true.

Tony And $j \cdot 5^n$ is $5j$ when $n = 1$, so if $j = 4$, $5j$ is 20. The second base case says it has to be 11. No good.

Answers

For You to Do

1. $$\begin{aligned} 2^n &= 7 \cdot 2^{n-1} - 10 \cdot 2^{n-2} \\ &= 7 \cdot 2^{n-1} - 5 \cdot 2 \cdot 2^{n-2} \\ &= 7 \cdot 2^{n-1} - 5 \cdot 2^{n-1} \\ &= 2 \cdot 2^{n-1} \\ &= 2^n \end{aligned}$$

2. $$\begin{aligned} 5^n &= 7 \cdot 5^{n-1} - 10 \cdot 5^{n-2} \\ &= 7 \cdot 5^{n-1} - 2 \cdot 5 \cdot 5^{n-2} \\ &= 7 \cdot 5^{n-1} - 2 \cdot 5^{n-1} \\ &= 5 \cdot 5^{n-1} \\ &= 5^n \end{aligned}$$

Enter Derman, who stares at the board.

Derman Why not do both? Do some 5^n's and some 2^n's. It would be $n \mapsto k \cdot 2^n + j \cdot 5^n$. Now you have two variables—I bet you can find k and j that make the function 4 at 0 and 11 at 1.

Sasha Two equations and two unknowns. I think he's on to something.

Tony Yes, but now it probably won't satisfy the recurrence.

Sasha Maybe it will. They each work by themselves. Look—take the two expressions and add down.

Sasha writes on the board.

$$\begin{array}{rcccc} & k \cdot 2^n & = & 7 \cdot k \cdot 2^{n-1} & - & 10 \cdot k \cdot 2^{n-2} \\ + & j \cdot 5^n & = & 7 \cdot j \cdot 5^{n-1} & - & 10 \cdot j \cdot 5^{n-2} \\ \hline & k \cdot 2^n + j \cdot 5^n & = & 7(k \cdot 2^{n-1} + j \cdot 5^{n-1}) & - & 10(k \cdot 2^{n-2} + j \cdot 5^{n-2}) \end{array}$$

Tony So, the sum of the two solutions will be another solution. Nice.

Derman See? I told you it would work.

Tony Don't worry, we believed you. Now we can find k and j to make the base cases work. Let's see—our closed form is $n \mapsto k \cdot 2^n + j \cdot 5^n$. If n is 0, we want this to be 4, so $k \cdot 2^0 + j \cdot 5^0 = 4$. That's just

$$k + j = 4$$

And when $n = 1$, the output is $k \cdot 2^1 + j \cdot 5^1$. We want this to be 11, so

$$2k + 5j = 11$$

Derman See? Two equations in two unknowns, like Sasha said. I told you it would work.

Sasha Fine, fine. Now we solve the system

$$\begin{aligned} k + j &= 4 \\ 2k + 5j &= 11 \end{aligned}$$

and we get $k = 3$ and $j = 1$. Looks like the closed form is

$$h(n) = 3 \cdot 2^n + 5^n$$

Tony I'll check it against the table . . . It worked!

Derman No doubt!

For You to Do

3. Show that Sasha, Tony and Derman's function h and the original function g agree for the domain of g. That is, show that the two functions

$$h(n) = 3 \cdot 2^n + 5^n \text{ and } g(n) = \begin{cases} 4 & \text{if } n = 0 \\ 11 & \text{if } n = 1 \\ 7g(n-1) - 10g(n-2) & \text{if } n > 1 \end{cases}$$

agree for all nonnegative integers n.

The natural domain of h is all of $\mathbb{R}$, while g is defined only for nonnegative integers.

Embedded in the above example is a general method for solving any two-term recurrence. You will give a precise description of the method in Exercise 6.

And Sasha's "add down" insight leads to another useful result.

Theorem 5.3 Closure of Solutions

If two functions r and s satisfy the two-term recurrence

$$f(n) = Af(n-1) + Bf(n-2)$$

then so does any linear combination

$$t(n) = k \cdot r(n) + j \cdot s(n)$$

where k and j are real numbers.

Remember...

A linear combination of x and y is $ax + by$, where a and b are real numbers.

Exercises Practicing Habits of Mind

Check Your Understanding

1. The two-term recurrence

$$f(n) = 7f(n-1) - 10f(n-2)$$

is satisfied by any function in the form $f(n) = A \cdot 2^n + B \cdot 5^n$. Each of the sequences below satisfies the recurrence. For each sequence, calculate the next two terms. Then find the values of A and B.

a. 4, 14, 58, . . .
b. 0, −3, −21, . . .
c. 4, 11, 37, . . .
d. 0, 1, 7, . . .
e. 1, 0, −10, . . .
f. **Take It Further** $x, y, -10x + 7y, \ldots$

Habits of Mind

Generalize. How can you use the other sequences in this exercise to help with part (f)?

Answers

For You to Do

3. $7(3 \cdot 2^{n-1} + 5^{n-1})$
$- 10(3 \cdot 2^{n-2} + 5^{n-2})$
$= (21 \cdot 2^{n-1} - 30 \cdot 2^{n-2})$
$+ (7 \cdot 5^{n-1} - 10 \cdot 5^{n-2})$
$= (21 \cdot 2^{n-1} - 15 \cdot 2^{n-1})$
$+ (7 \cdot 5^{n-1} - 2 \cdot 5^{n-1})$
$= 6 \cdot 2^{n-1} + 5 \cdot 5^{n-1}$
$= 3 \cdot 2^n + 5^n$
$= h(n)$

Exercises

1. a. 266 and 1282; $A = 2$, $B = 2$
b. −117 and −609; $A = 1$, $B = -1$
c. 149 and 673; $A = 3$, $B = 1$
d. 39 and 203; $A = -\frac{1}{3}$, $B = \frac{1}{3}$
e. −70 and −390; $A = \frac{5}{3}$, $B = -\frac{2}{3}$
f. $-70x + 39y$ and $-390x + 203y$; $A = \dfrac{5x - y}{3}$, $B = \dfrac{y - 2x}{3}$

Explore

For You to Do

PROBLEM 3 You can verify the base cases $n = 0$ and $n = 1$ directly. The induction step looks very similar to what Sasha drew in the dialog, using the specific numbers $k = 3$ and $j = 1$. The goal is to show that $h(n) = 7h(n-1) - 10h(n-2)$ for $n \geq 2$; in other words, show that

$$3 \cdot 2^n + 5^n = 7(3 \cdot 2^{n-1} + 5^{n-1}) - 10(3 \cdot 2^{n-2} + 5^{n-2})$$

Students were asked to do similar work in part (b) of Exercise 3 in Lesson 5.12, and are asked in the upcoming exercises to do this type of work.

Wrap Up

Give students a two-term recurrence they have not seen before, such as

$$f(n) = 13f(n-1) - 36f(n-2)$$

with initial conditions $f(0) = 5$, $f(1) = 35$, and have them construct the closed form

$$f(n) = 2 \cdot 4^n + 3 \cdot 9^n$$

Or explore the Fibonacci closed form (Exercise 7).

Assessment Resources

Lesson Quiz 5.13

1. Find a closed-form definition for a function that agrees with h.

$$h(a) = \begin{cases} 4 & \text{if } a = 0 \\ 14 & \text{if } a = 1 \\ 7h(a-1) - 12h(a-2) & \text{if } a > 1 \end{cases}$$

2. Consider the two-term recurrence

$$f(n) = 11f(n-1) - 18f(n-2)$$

a. Show that 2^n satisfies the recurrence.
b. Find a closed-form definition for f if the initial conditions are $f(0) = 6$ and $f(1) = 33$.

Exercises

HOMEWORK

- Core: 8, 9, 10, 12, 16, 17
- Optional: 11
- Extension: 13, 14, 15

Check Your Understanding

EXERCISE 1 Look for any student solving the general case first, rather than repeatedly solving the 2-by-2 system of equations.

EXERCISE 2 The proofs here should help students understand the connection between quadratics and the corresponding solutions to recurrences. The sum and product results are fundamental to the proofs and explain why both 10^n and 3^n can solve the recurrence.

EXERCISE 7 is the key exercise of the lesson; students are being asked to generate the Binet formula for Fibonacci numbers. Students who understand the prior exercises, especially the ones which do not factor, may have immediate success with this.

If you feel your students are not ready, assign the On Your Own exercises, then return to do this one in class. This exercise is definitely one to give students a chance to solve; those who do should be very satisfied with what they have accomplished!

GOING FURTHER You might also point out that this exercise explains why the ratio of successive Fibonacci numbers approaches the golden ratio. As n gets higher, the second term approaches zero since $\frac{1-\sqrt{5}}{2}$ is between -1 and 1, so $f(n)$ gets closer and closer to the powers of the golden ratio.

2. Consider the two-term recurrence

$$g(n) = 13g(n-1) - 30g(n-2)$$

a. What two numbers have a sum of 13 and a product of 30?

b. Show that $g(n) = 10^n$ satisfies the recurrence.

c. Show that $g(n) = 3^n$ satisfies the recurrence.

d. Show that $g(n) = 10^n + 3^n$ satisfies the recurrence.

That is, show that $10^n = 13 \cdot 10^{n-1} - 30 \cdot 10^{n-2}$ for any integer $n \geq 2$.

3. Find a closed-form definition for a function that agrees with t.

$$t(n) = \begin{cases} 5 & \text{if } n = 0 \\ 19 & \text{if } n = 1 \\ 8t(n-1) - 15t(n-2) & \text{if } n > 1 \end{cases}$$

Habits of Mind

Make a connection. What connection is there between this and $x^2 - 10x + 21$?

4. Define function t by

$$t(n) - 10t(n-1) + 21t(n-2) = 0$$

with $t(0) = 2$ and $t(1) = 10$. Find a closed-form equivalent for t.

5. Find different exponential functions $h(n) = a^n$ and $j(n) = b^n$ that satisfy the two-term recurrence

$$f(n) = 2f(n-1) + f(n-2)$$

Use the quadratic formula.

6. **Write About It** Describe a method for finding a closed form (in terms of A and B) for a function that satisfies the two-term recurrence

$$f(n) = Af(n-1) + Bf(n-2)$$

7. The Fibonacci numbers are the outputs of the function f defined by

$$f(n) = f(n-1) + f(n-2)$$

with the initial conditions $f(0) = 0$ and $f(1) = 1$. Find a closed-form equivalent for f.

On Your Own

8. Function f satisfies the recurrence

$$f(n) = 9f(n-1) - 20f(n-2)$$

Find a closed-form equivalent for f if

a. $f(0) = 2$ and $f(1) = 9$

b. $f(0) = 20$ and $f(1) = 90$

Answers

2. a. 10 and 3

b.
$$\begin{aligned} & 13 \cdot 10^{n-1} - 30 \cdot 10^{n-2} \\ &= 13 \cdot 10^{n-1} - 3 \cdot 10 \cdot 10^{n-2} \\ &= 13 \cdot 10^{n-1} - 3 \cdot 10^{n-1} \\ &= 10 \cdot 10^{n-1} \\ &= 10^n \end{aligned}$$

c.
$$\begin{aligned} & 13 \cdot 3^{n-1} - 30 \cdot 3^{n-2} \\ &= 13 \cdot 3^{n-1} - 10 \cdot 3 \cdot 3^{n-2} \\ &= 13 \cdot 3^{n-1} - 10 \cdot 3^{n-1} \\ &= 3 \cdot 3^{n-1} \\ &= 3^n \end{aligned}$$

d.
$$\begin{aligned} & 13(10^{n-1} + 3^{n-1}) - 30(10^{n-2} + 3^{n-2}) \\ &= (13 \cdot 10^{n-1} - 30 \cdot 10^{n-2}) + \\ & \quad (13 \cdot 3^{n-1} - 30 \cdot 3^{n-2}) \end{aligned}$$

See proofs in parts (b) and (c) to complete the proof.

3. $t(n) = 3 \cdot 3^n + 2 \cdot 5^n$

4. $t(n) = 3^n + 7^n$

5. $h(n) = (1 + \sqrt{2})^n$, $j(n) = (1 - \sqrt{2})^n$

6. Answers may vary. Sample: Use the quadratic equation $x^2 = Ax + B$ to find the two roots that are used to determine the closed form. Either root of the equation $x^2 - Ax - B = 0$ can be the base.

7. $f(n) = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$

8. a. $f(n) = 4^n + 5^n$

b. $f(n) = 10(4^n + 5^n)$

9. Consider the two-term recurrence $f(n) = 8f(n-1) - 12f(n-2)$.

a. Why might you guess that an exponential function would satisfy the recurrence?

b. Show that $f(n) = 6^n$ satisfies this recurrence.

c. Find a closed-form definition for f if the initial conditions are $f(0) = 2$ and $f(1) = 8$.

d. Find a closed-form definition for f if the initial conditions are $f(0) = 5$ and $f(1) = 26$.

10. Find a closed-form equivalent for a function f that fits this table by first determining a two-term recurrence that the function satisfies.

x	$f(x)$
0	2
1	10
2	36
3	152
4	592
5	2400

11. Suppose f satisfies the recurrence $f(n) = 6f(n-1) - 7f(n-2)$.

a. Show that if $f(0) = 1$ and $f(1) = 3 + \sqrt{2}$, then $f(2) = (3 + \sqrt{2})^2$.

b. Find both possible values of b that make $f(n) = b^n$ satisfy the recurrence.

c. If $f(0) = 2$ and $f(1) = 6$, find a closed-form definition for f.

d. If $f(0) = 0$ and $f(1) = 1$, find a closed-form definition for f.

12. Find a closed-form definition for the function $L(n) = L(n-1) + L(n-2)$ with $L(0) = 2$ and $L(1) = 1$. This function generates the Lucas numbers

$$2, 1, 3, 4, 7, 11, \ldots$$

13. **Take It Further** Define function f by

$$f(n) = 4f(n-1) - 13f(n-2)$$

with $f(0) = 2$ and $f(1) = 4$.

a. Tabulate f using inputs from 0 to 8.

b. Find a closed-form equivalent for f using the techniques from this lesson. What is different about this exercise? Does the process still work?

14. **Take It Further** Suppose f satisfies the recurrence

$$f(n) = 19f(n-2) - 30f(n-3)$$

Find a closed-form definition for f if $f(0) = 8$, $f(1) = 3$, and $f(2) = 79$.

Go Online PHSchool.com

For additional practice, go to **Web Code:** bga-0513

On Your Own

EXERCISE 10 You might remind students that there is more than one possible function f that fits the table, but the two-term recurrence leads to a specific closed-form definition.

This is a fairly difficult exercise, since the recurrence is unknown. Consider asking students to determine the recurrence as a separate exercise.

EXERCISE 11 The solution method here is similar to Exercise 7 on the Fibonacci numbers.

EXERCISE 14 requires students to think about how they can extend the rules for two-term recurrences. In general, closed forms for three-term recurrences are difficult to find due to the difficulty of solving cubic equations. Here, the equation has three integer roots, so students can find the form more directly.

9. a. Answers may vary. Sample: One way would be to tabulate $f(n)$ and notice that a common ratio between terms emerges, suggesting that this ratio might fit the recurrence.

b. $8 \cdot 6^{n-1} - 12 \cdot 6^{n-2}$
$= 8 \cdot 6^{n-1} - 2 \cdot 6 \cdot 6^{n-2}$
$= 8 \cdot 6^{n-1} - 2 \cdot 6^{n-1}$
$= 6 \cdot 6^{n-1}$
$= 6^n$

c. $f(n) = 6^n + 2^n$

d. $f(n) = 4 \cdot 6^n + 2^n$

10. $f(n) = \frac{7}{3} \cdot 4^n - \frac{1}{3} \cdot (-2)^n$

11. a. $f(2) = 6(3 + \sqrt{2}) - 7(1)$
$= 11 + 6\sqrt{2}$
$= (3 + \sqrt{2})^2$

b. $b_1 = 3 + \sqrt{2}$, $b_2 = 3 - \sqrt{2}$

c. $f(n) = (3 + \sqrt{2})^n + (3 - \sqrt{2})^n$

d. $f(n) = \frac{1}{2\sqrt{2}} \cdot ((3 + \sqrt{2})^n - (3 - \sqrt{2})^n)$

12. $L(n) = \left(\frac{1 + \sqrt{5}}{2}\right)^n + \left(\frac{1 - \sqrt{5}}{2}\right)^n$

13 a.

n	$f(n)$
0	2
1	4
2	−10
3	−92
4	−238
5	244
6	4070
7	13,108
8	−478

b. $f(n) = (2 + 3i)^n + (2 - 3i)^n$
The process still works, but the difference is that $f(n)$ uses complex numbers.

14. $f(n) = 5 \cdot 2^n + 3^n + 2 \cdot (-5)^n$

EXERCISE 15 closes a loophole, addressing the issue of a double root in the quadratic characteristic polynomial. The solution is similar to how you deal with a double root in the method of partial fractions.

Maintain Your Skills

EXERCISE 17 may be a review of material students learned in an earlier course, so use this exercise as a gauge to see if students need a refresher.

Additional Resources

PRINT RESOURCES
- Solution Manual
- Practice Workbook
- Assessment Resources
- Teaching Resources

TECHNOLOGY
- Interactive Textbook
- TeacherExpress CD-ROM
- ExamView CD-ROM
- PHSchool.com

Additional Practice

1. The two-term recurrence

$$f(n) = 4f(n-1) - 3f(n-2)$$

is satisfied by any function in the form $f(n) = A \cdot 1^n + B \cdot 3^n$. Each of the sequences below satisfies the recurrence. For each sequence, calculate the next two terms. Then find the values of A and B.

a. 5, 11, 29, . . . **b.** 3, 5, 11, . . .
c. 10, 18, 42, . . . **d.** −2, 2, 14, . . .

2. Function g satisfies the recurrence

$$g(n) = 8g(n-1) - 12g(n-2)$$

Find a closed-form equivalent for g if
a. $g(0) = 2$ and $g(1) = 8$ **b.** $g(0) = 10$ and $g(1) = 40$

3. Consider the two-term recurrence

$$f(n) = 9f(n-1) - 18f(n-2)$$

a. What two numbers have a sum of 9 and a product of 18?
b. Show that $f(n) = 3^n$ satisfies the recurrence.
c. Show that $f(n) = 6^n$ satisfies the recurrence.

4. Nicholas invests $600 every year in a savings account. Each year he also earns 4% on the money currently invested. Find a closed-form definition for a function that gives the amount of money in Nicholas's account after n years.

5. Michaela wants to save money to buy a house. She estimates she can save $7000 a year toward a down payment.
a. How much can Michaela expect to have in her account in three years if the money in her account grows by 7% each year?
b. How much more would Michaela have in her account in three years at 10% growth each year instead of 7%?

6. a. Find the correct monthly payment on a $12,000 car loan taken at 8% APR for 36 months.
b. Find the correct monthly payment on a $22,000 car loan taken at 9% APR for 48 months.

Practice: For Lesson 5.13, assign Exercises 1–3.

15. Take It Further There is a special case of recurrences in the form $f(n) = Af(n-1) + Bf(n-2)$. One such recurrence is

$$f(n) = 2f(n-1) - f(n-2)$$

a. Describe the function with starting conditions $f(0) = 3$ and $f(1) = 8$.

b. Describe the function with starting conditions $f(0) = 10$ and $f(1) = 0$.

c. Describe the function in general for all functions that satisfy this recurrence.

d. Describe the function in general for all functions that satisfy the similar recurrence.

$$f(n) = 6f(n-1) - 9f(n-2)$$

16. Standardized Test Prep Consider the following function.

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ f(n-1) + f(n-2) & \text{if } n > 1 \end{cases}$$

Which statement is correct?

A. $f(2n) = 2(f(n-1) + f(n-2))$

B. $f(2n) = f(2(n-1)) + f(2(n-2))$

C. $f(2n) = f(2n-1) + f(2n-2)$

D. $f(2n) = f(2n-2) + f(2n-3)$

Maintain Your Skills

17. The Martinez family buys a house. They get a 15-year mortgage at a low rate of 6% APR. This means that what they owe at the end of the year is 6% more than what they owed at the start of the year. After the interest is added, the family reduces what they owe by making a payment. The family owes $200,000 to start.

a. If the Martinez family pays $12,000 every year, explain why the function B gives their balance after n years.

$$B(n) = \begin{cases} 200{,}000 & \text{if } n = 0 \\ 1.06B(n-1) - 12{,}000 & \text{if } n > 0 \end{cases}$$

For simplicity, calculate interest and payment once per year, rather than once per month.

b. If the Martinez family pays $12,000 every year, will they pay off the mortgage in 15 years? If not, how much is left at the end? What if they pay $13,000? $14,000? $15,000? Look for a pattern.

c. If the family pays D dollars every year, how much is left at the end in terms of D?

d. Determine the exact amount the Martinez family should pay per year to leave a $0 balance on the mortgage at the end.

Answers

15. a. Outputs follow the form $f(n) = 5n + 3$.
b. Outputs follow the form $f(n) = -10n + 10$.
c. Any function in the form $f(n) = pn + q$ satisfies the recurrence.
d. The general closed form is $f(n) = (An + B) \cdot 3^n$.

16. C

17. a. The initial balance would be $200,000. The 1.06 term adds the 6% interest and the $12,000 payment is deducted each year.
b. No; $200,000; no; $176,724 no; $153,448; no; $130,172. The value after 15 years appears to be dropping linearly.
c. $B = 479{,}312 - 23.276D$
d. $20,592.55

5.14 $f(n) = Af(n-1) + B$

The right strategy for solving a recurrence depends on the form of the recurrence. Lesson 5.13 showed how combining two exponential solutions to a recurrence of the form $f(n) = Af(n-1) + Bf(n-2)$ produces yet another solution. In this lesson, you will draw on skills developed earlier to solve recurrences of the form $f(n) = Af(n-1) + B$.

Minds in Action

episode 17

Sasha and Tony are looking at the newspaper.

Tony All these car ads make me think of the monthly payment function we figured out last year.

Sasha If you think about it, we were solving a recurrence. Remember this problem?

> Suppose you want to buy a car that costs \$10,000. You can put \$1000 down, so you would borrow \$9000. The interest rate is 5%. The dealer wants the loan paid off in three years. Will a monthly payment of \$250 pay off the loan?

Tony Yes, I remember what we did. We figured out a function b that gave us the balance at the end of month n with a monthly payment of 250 dollars. Then we figured out $b(36)$ and saw if it was 0. The function definition was recursive:

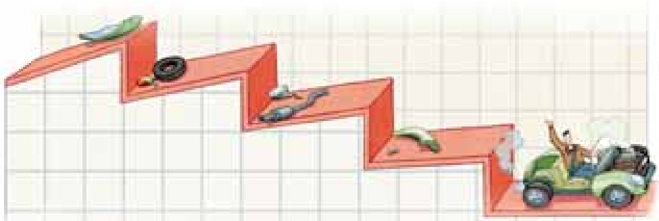
$$b(n) = \begin{cases} 9000 & \text{if } n = 0 \\ \left(1 + \frac{0.05}{12}\right)b(n-1) - 250 & \text{if } n > 0 \end{cases}$$

We didn't get 0, but then we adjusted the monthly payment to make $b(36)$ come out to be 0.

Sasha And later, we unstacked this and got a formula for the monthly payment. But wait. If we think about it in terms of recurrence relations, the recurrence looks like

$$b(n) = Ab(n-1) + B$$

Maybe we can use what we did for monthly payments to solve any recurrence of this form.



Remember...

What you owe at the end of the month is what you owed at the start of the month plus $\frac{1}{12}$th of the year's interest on that amount, minus the monthly payment.

What is the actual monthly payment?

Lesson Overview

GOAL

- Calculate the monthly payment for a car loan and write a general rule for all such problems.

This lesson revisits the monthly-payment problem as a recurrence in the form $f(n) = Af(n-1) + B$, solving the general case. The problems present multiple paths to the result, including a path that looks at how far away outputs are from an equilibrium point, a value of $f(n)$ that repeats. Relative to the equilibrium point, these functions are purely exponential.

CHECK YOUR UNDERSTANDING
- Core: 1, 2, 3
- Optional: 4, 5
- Extension: 6

MATERIALS
- CAS (recommended)
- graphing calculators

HOMEWORK
- Core: 7, 9, 10, 11, 14
- Optional: 8, 15
- Extension: 12, 13

VOCABULARY
- equilibrium point

Launch

Review Exercise 17 from the previous lesson, or launch directly into the dialog that starts the lesson.

Explore

Developing Habits of Mind

Skip the language of affine transformations unless students covered this material in CME Project *Algebra 2*. Students have used and will continue to use the geometric series expansion, which is a good way of understanding the resulting closed form given as Theorem 5.4.

If students are having trouble, take them through a numeric example without evaluating; they will see which numbers they can combine in what ways.

For Discussion

PROBLEM 1 The last step in the Developing Habits of Mind gives

$f(n) = A^n f(0) + B(A^{n-1} + A^{n-2} + \cdots + A + 1)$

Now use the formula for the sum of a geometric series to condense the sum.

$$f(n) = A^n f(0) + B\left(\frac{A^n - 1}{A - 1}\right)$$

Finally, note that $f(0) = p$ is given as a base case, so

$$f(n) = A^n \cdot p + B\left(\frac{A^n - 1}{A - 1}\right)$$

Developing Habits of Mind

Consider more than one strategy. There are two methods from CME Project *Algebra 2* to solve the recurrence

$$f(n) = Af(n-1) + B$$

- You could use affine transformations. If $\mathcal{A}_{(a,b)}(x) = ax + b$, then the recurrence is

$$f(n) = \mathcal{A}_{(A,B)}(f(n-1))$$

But then

$$f(n-1) = \mathcal{A}_{(A,B)}(f(n-2))$$

and

$$f(n-2) = \mathcal{A}_{(A,B)}(f(n-3))$$

and so on, so that

$$f(n) = \mathcal{A}^n_{(A,B)}(f(0))$$

where $\mathcal{A}^n_{(A,B)}$ is the nth iteration of $\mathcal{A}_{(A,B)}$. In CME Project *Algebra 2*, you derived a formula for the nth iteration of an affine transformation.

- If you did not want to use the language of affine transformations, you could just unstack the recurrence:

$$\begin{aligned} f(n) &= Af(n-1) + B \\ &= A(Af(n-2) + B) + B = A^2 f(n-2) + AB + B \\ &= A^2(Af(n-3) + B) + AB + B = A^3 f(n-3) + A^2B + AB + B \\ &\vdots \\ &= A^n f(0) + B(A^{n-1} + A^{n-2} + \cdots + A + 1) \end{aligned}$$

The rightmost sum is a geometric series. You know a formula for that.

Solving a recurrence means finding a function that satisfies the recurrence at every input in the function's domain.

For Discussion

1. Finish one or both of these derivations to prove the following theorem.

Theorem 5.4 ***Closed-form equivalent for $f(n) = Af(n-1) + B$***

Define f by

$$f(n) = \begin{cases} p & \text{if } n = 0 \\ Af(n-1) + B & \text{if } n > 0 \end{cases}$$

Then for nonnegative integer inputs, a closed-form equivalent for f is

$$f(n) = A^n \cdot p + B\left(\frac{A^n - 1}{A - 1}\right)$$

Here, p is a number, the base case.

Answers

For Discussion

1. $f(n) = A^n f(0) + B(A^{n-1} + A^{n-2} + \cdots + A + 1)$
Using the formula for the sum of a geometric series to condense the sum: $f(n) = A^n \cdot p + B\left(\frac{A^n - 1}{A - 1}\right)$.

For You to Do

Tristen buys a car. The car costs \$12,000. The financing is 6% APR, or 0.5% monthly. A car payment is due every month for 36 months.

2. Find Tristen's car payment, to the nearest cent.
3. Using the same term and interest rate, construct a function m so that $m(C)$ is the monthly payment for a car that costs C dollars.

How could a CAS help you do this without guesswork?

Exercises *Practicing Habits of Mind*

Check Your Understanding

1. Jess invests \$500 every year in a savings account. Each year she also earns 3% on the money currently invested. Find a closed-form definition for a function that gives the amount of money in Jess's account after n years.
2. Generalize the result from Exercise 1. Suppose Jess invests D dollars per year in a savings account that earns a rate of r per year (as a decimal, not a percent). Find a function B that gives the balance of Jess's account after n years.
3. A local environmental group claims that 20% of the trees in Woodville are being cut down each year. Authorities have decided to plant 3000 trees every year to try and counteract this effect.
   a. Suppose there are 25,000 trees in Woodville to start. Determine how many trees there will be each year for the next seven years, rounding to the nearest tree.
   b. Make a scatter plot with number of trees on the vertical axis and years on the horizontal axis.
   c. Find an *equilibrium point*, the number of trees that will remain constant year-to-year after a certain number of iterations of this process.
   d. How many more trees than the equilibrium point does Woodville have at the start? After one year, how many more trees than the equilibrium point are there? After two years? After three? Look for a pattern.
   e. Find a closed-form definition for $T(n)$, the number of trees in Woodville after n years.

An **equilibrium point** is a repeating value of an iteration. If $f(n) = f(n-1) = C$, then C is an equilibrium point.

For You to Do

2. \$365.06
3. $m(C) = \dfrac{1.005^n \cdot C \cdot 0.005}{1.005^n - 1}$

Exercises

1. $B(n) = 500\left(\dfrac{1.03^n - 1}{0.03}\right)$
2. $B(n) = D\left(\dfrac{(1+r)^n - 1}{r}\right)$

3. a.

Years, n	Trees, $T(n)$
0	25,000
1	23,000
2	21,400
3	20,120
4	19,096
5	18,277
6	17,621
7	17,097

b–e. See back of book.

For You to Do

PROBLEM 3 Use the result from Theorem 5.4, and set the balance equal to zero for payment m and initial balance C dollars:

$$0 = (1.005)^n \cdot C - m\left(\frac{1.005^n - 1}{0.005}\right)$$

Then solve for m in terms of C.

Note that $m(C)$ is directly proportional to C, so (for example) a car twice as expensive has twice the monthly payment.

Wrap Up

Review any methods students have used to solve the monthly payment problem or the general recurrence.

Assessment Resources

Lesson Quiz 5.14

1. Katie invests \$1000 every year in a savings account. Each year she also earns 4% interest on the money currently invested. Find a closed-form definition for a function that gives the amount of money in Katie's account after n years.
2. Jack takes out a loan for \$20,000 with 6% yearly interest compounded monthly. He will pay it off in monthly installments of \$500.
   a. Determine how much Jack will owe each month for the first year.
   b. Find a closed-form definition for $P(n)$, the amount of money Jack owes after n months.
3. Define the function f by the recurrence

$$f(n) = 1.075f(n-1) + C$$

with $f(0) = 325{,}000$. Find C if 325,000 is the equilibrium point for f, the value of $f(n)$ that produces a constant output.

Exercises

HOMEWORK

- Core: 7, 9, 10, 11, 14
- Optional: 8, 15
- Extension: 12, 13

Check Your Understanding

EXERCISE 1 If students have trouble with this exercise, suggest that they write the expression for $B(1)$, $B(2)$, and so on without actually computing the balance. This will help them see the pattern.

EXERCISE 2 If students get stuck here, encourage them to follow another numeric example in the style of Exercise 1. If students can describe how the work is done there, they should be able to generalize.

Note that this result is used in Exercise 7, so review this exercise before assigning the homework.

EXERCISE 3 This perspective (comparing to an equilibrium point) can help students quickly find closed forms for these situations, as the function is exponential relative to the equilibrium point.

EXERCISE 6 The algebra of this exercise is not overly challenging; what is challenging is the concept that each term is A (here, $1 + r$) times as far from the equilibrium point as the term before it. The equilibrium value is simple to calculate, and students have calculated this at least twice before (including Exercise 4).

On Your Own

EXERCISE 7 One purpose here is for students to recognize that over the long term, the effect of these rate changes is more than just a proportional percentage. The change from 10% to 12% gives more than a proportional (20%) change in the resulting balance.

4. Define function f by the recurrence

$$f(n) = 1.065f(n-1) + B$$

with $f(0) = 200{,}000$. Find B if 200,000 is the equilibrium point for f, the value of $f(0)$ that produces a constant output.

5. Show that the equilibrium point for the recurrence

$$f(n) = Af(n-1) + B$$

is

$$n = \frac{B}{1-A}$$

6. **Take It Further** The general form for a car payment or mortgage comes from the recurrence

$$f(n) = (1+r)f(n-1) - P$$

Given $f(0) = B$ (the starting balance), the goal is to find the correct value of P so that $f(a) = 0$ where a is the length of the loan and r is the interest rate expressed as a decimal.

   a. Find the equilibrium point of f in terms of P and r.
   b. How far is the starting balance B from the equilibrium point?
   c. How far away from the equilibrium point is $f(1)$? $f(2)$? $f(n)$?
   d. Write a closed-form definition for f in terms of B, P, r, and n.
   e. Solve the equation $f(n) = 0$ to find a function for the monthly payment P in terms of the initial balance B, the rate r, and the number of payments n.

On Your Own

7. At 30 years old, Drew estimates he can save \$5000 per year toward retirement. Use the result from Exercise 2 to answer these questions.

   a. How much can Drew expect to have in his retirement account at age 65 if he assumes that each year, the money in his account will grow by 10%?
   b. How much more would Drew have in his retirement account at age 65 at 12% growth instead of 10%?
   c. How much less would Drew have in his retirement account at age 65 at 8% growth instead of 10%?

10% growth per year is a reasonable average when investing in commodities such as stocks. The actual percentage varies year-to-year but a fixed percentage is usually used when estimating long-term results like this one.

Answers

4. $B = -13{,}000$

5. If x is an equilibrium point, then $f(n) = x$ for all n. Therefore:

$x = Ax + B$
$x - Ax = B$
$x(1 - A) = B$
$x = \frac{B}{1-A}$

6. a. $\frac{P}{r}$

b. $B - \frac{P}{r}$

c. $f(1)$ is $(1+r)\left(B - \frac{P}{r}\right)$ away from the equilibrium point.

$f(2)$ is $(1+r)^2\left(B - \frac{P}{r}\right)$ away from the equilibrium point.

$f(n)$ is $(1+r)^n\left(B - \frac{P}{r}\right)$ away from the equilibrium point.

d. $f(n) = \frac{P}{r} + (1+r)^n\left(B - \frac{P}{r}\right)$

e. $P = \frac{Br(1+r)^n}{(1+r)^n - 1}$

7. a. \$1,355,121.84

b. \$803,195.64

c. \$493,537.82

8. Define f recursively as

$$f(n) = \begin{cases} P & \text{if } n = 0 \\ Af(n-1) + B & \text{if } n > 0 \end{cases}$$

a. Tabulate f for inputs from 0 to 4.

b. Find a closed-form equivalent for f.

9. Consider Tristen's situation with a \$12,000 car loan at 6% APR for 36 months. The recursive definition for Tristen's balance is

$$B(n, p) = \begin{cases} 12{,}000 & \text{if } n = 0 \\ 1005B(n-1, p) - p & \text{if } n > 0 \end{cases}$$

a. Use the results from the previous exercise to write $f(36)$ in terms of p, the monthly payment.

b. Find the correct monthly payment to the nearest cent.

If 6% is the APR, what is the monthly interest rate?

10. **a.** Find the correct monthly payment on a \$10,000 car loan taken at 9% APR for 48 months,

b. Find the correct monthly payment on a \$20,000 car loan with the same terms. What happens to the monthly payment?

11. **Write About It** Prove the following theorem.

Theorem 5.5 *Monthly Payments*

On a loan for C dollars taken out for n months with an APR of i percent compounded monthly, the monthly payment is m dollars where

$$m = \frac{q^n(q-1)}{q^n - 1}C$$

and $q = 1 + \frac{i}{1200}$.

12. **Take It Further** In this investigation you explored two-term recurrences in the form

$$f(n) = Af(n-1) + Bf(n-2)$$

A related form is

$$f(n) = Af(n-1) + Bf(n-2) + Cf(n-3)$$

Consider the three-term recurrence

$$f(n) = 10f(n-1) - 31f(n-2) + 30f(n-3)$$

a. Show that $f(n) = 2^n$ satisfies this recurrence.

b. Find two other exponential functions that satisfy the recurrence.

c. If $f(0) = 4$, $f(1) = 16$, and $f(2) = 74$, find a closed-form definition for f.

EXERCISE 9 provides a second method of solving the monthly payment problem, which is different from the method seen in the Check Your Understanding exercises that relies on the equilibrium point. Both methods are valuable and lead to the same result.

ERROR PREVENTION Watch out for students making the mistake of letting $B = p$ instead of $B = -p$.

EXERCISE 10 If students work on the general form here, have them explain why the general form implies the doubling behavior seen in this exercise.

8. a.

n	$f(n)$
0	P
1	$AP + B$
2	$A^2P + AB + B$
3	$A^3P + A^2B + AB + B$
4	$A^4P + A^3B + A^2B + AB + B$

b. $f(n) = A^nP + B\left(\frac{A^n - 1}{A - 1}\right)$

9. a. $f(36) = 1.005^{36} \cdot 12{,}000 - p\left(\frac{1.005^{36} - 1}{0.005}\right)$

b. \$365.06

10. a. \$248.85

b. \$497.70; it doubles.

11. $f(n) = pA^n + B\left(\frac{A^n - 1}{A - 1}\right)$, $A = q$, $p = C$, $m = -b$

$f(n) = Cq^n - m\left(\frac{q^n - 1}{q - 1}\right)$

The monthly payment is determined at $f(n) = 0$.

$0 = Cq^n - m\left(\frac{q^n - 1}{q - 1}\right)$

$m\left(\frac{q^n - 1}{q - 1}\right) = Cq^n$

$m = \frac{Cq^n(q-1)}{q^n - 1}$

12. a.
$$\begin{aligned} & 10 \cdot 2^{n-1} - 31 \cdot 2^{n-2} + 30 \cdot 2^{n-3} \\ &= 10 \cdot 2^{n-1} - 31 \cdot 2^{n-2} + 15 \cdot 2^{n-2} \\ &= 10 \cdot 2^{n-1} - 16 \cdot 2^{n-2} \\ &= 10 \cdot 2^{n-1} - 8 \cdot 2^{n-1} \\ &= 2 \cdot 2^{n-1} \\ &= 2^n \end{aligned}$$

b. $f(n) = 3^n$ and $f(n) = 5^n$

c. $f(n) = 2 \cdot 2^n - 3^n + 3 \cdot 5^n$

Additional Resources

PRINT RESOURCES
- Solution Manual
- Practice Workbook
- Assessment Resources
- Teaching Resources

TECHNOLOGY
- Interactive Textbook
- TeacherExpress CD-ROM
- ExamView CD-ROM
- PHSchool.com

Additional Practice

1. The two-term recurrence

$$f(n) = 4f(n-1) - 3f(n-2)$$

is satisfied by any function in the form $f(n) = A \cdot 1^n + B \cdot 3^n$. Each of the sequences below satisfies the recurrence. For each sequence, calculate the next two terms. Then find the values of A and B.

a. 5, 11, 29, . . .
b. 3, 5, 11, . . .
c. 10, 18, 42, . . .
d. −2, 2, 14, . . .

2. Function g satisfies the recurrence

$$g(n) = 8g(n-1) - 12g(n-2)$$

Find a closed-form equivalent for g if
a. $g(0) = 2$ and $g(1) = 8$
b. $g(0) = 10$ and $g(1) = 40$

3. Consider the two-term recurrence

$$f(n) = 9f(n-1) - 18f(n-2)$$

a. What two numbers have a sum of 9 and a product of 18?
b. Show that $f(n) = 3^n$ satisfies the recurrence.
c. Show that $f(n) = 6^n$ satisfies the recurrence.

4. Nicholas invests \$600 every year in a savings account. Each year he also earns 4% on the money currently invested. Find a closed-form definition for a function that gives the amount of money in Nicholas's account after n years.

5. Michaela wants to save money to buy a house. She estimates she can save \$7000 a year toward a down payment.
a. How much can Michaela expect to have in her account in three years if the money in her account grows by 7% each year?
b. How much more would Michaela have in her account in three years at 10% growth each year instead of 7%?

6. a. Find the correct monthly payment on a \$12,000 car loan taken at 8% APR for 36 months.
b. Find the correct monthly payment on a \$22,000 car loan taken at 9% APR for 48 months.

Practice: For Lesson 5.14, assign Exercises 4–6.

13. **Take It Further** Find a three-term recurrence that this sequence satisfies. Then find a closed-form definition for a function that satisfies the recurrence and agrees with the sequence.

1, 5, 105, 965, 10065, 99725, 1000665

14. **Standardized Test Prep** Find, to the nearest penny, the monthly payment on a \$12,000 loan at 6% APR, paid off over 36 months.

A. \$362.97 **B.** \$365.06 **C.** \$368.44 **D.** \$370.60

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For additional practice, go to **Web Code:** bga-0514

Maintain Your Skills

15. Consider the recurrence

$$f(n) = 3f(n-1) - 300$$

a. Show that 150 is an equilibrium point for f. That is, if $f(n-1) = 150$, then $f(n) = 150$.
b. Suppose $f(0)$ is 1000 greater than the equilibrium point. How much greater than the equilibrium point is $f(1)$? $f(2)$? $f(3)$? $f(n)$?

What is the monthly payment on a \$12,000 car loan at 6% APR, paid off over 36 months?

Answers

13. $$f(n) = \begin{cases} 1 & n = 0 \\ 5 & n = 1 \\ 9f(n-1) + 16f(n-2) - 60f(n-3) & n > 1 \end{cases}$$

$F(n) = 10^n + (-3)^n - 2^n$

14. B

15. a. Calculate directly: if $f(n-1) = 150$ then $f(n) = 3 \cdot 150 - 300 = 150$.
b. $f(1)$ is 3000 greater than the equilibrium point.
$f(2)$ is 9000 greater than the equilibrium point.
$f(3)$ is 27,000 greater than the equilibrium point.
$f(n)$ is $1000 \cdot 3^n$ greater than the equilibrium point.